Analysis I Sheet 4 MT05

Algebra of limits, subsequences

1 For which of the following a_n does the sequence (a_n) converge? Find the value of the limit when it exists.

$$\text{(i)}\quad \frac{3n^3+n^2+1}{2n^3-100n-3} \qquad \text{(ii)}\quad \frac{2^n n^2+3^n n}{3^n(n+1)+n^7} \qquad \text{(iii)}\quad \sqrt{n+1}-\sqrt{n}$$

2 Let $z_n = a_n + \mathrm{i}b_n$ and $w_n = c_n + \mathrm{i}d_n$ where $a_n, b_n, c_n, d_n \in \mathbb{R}$. Assuming that $w_n \neq 0$, find $\mathrm{Re}(z_n/w_n)$ and $\mathrm{Im}(z_n/w_n)$ in terms of a_n, b_n, c_n, d_n.

Using the Algebra of Limits *for real sequences*, prove that if $z_n \to \alpha$ and $w_n \to \beta \neq 0$, then $z_n/w_n \to \alpha/\beta$.

3 Let (a_n) be a sequence of real numbers. Define three new sequences (u_n) and (v_n) by setting $u_n := a_{2n+1}$, $v_n := a_{2n}$, $w_n := a_{3n}$. Prove carefully from the definitions that:

(i) if $a_n \to \ell$ as $n \to \infty$ then $u_n \to \ell$, $v_n \to \ell$, $w_n \to \ell$ as $n \to \infty$;
(ii) if $u_n \to \ell$ and $v_n \to \ell$ as $n \to \infty$ then $a_n \to \ell$ as $n \to \infty$;
(iii) if $u_n \to p$, $v_n \to q$, and $u_n \to r$ as $n \to \infty$ then $p = r$, $q = r$ and hence $a_n \to p$ as $n \to \infty$.

Give an example of a divergent sequence (c_n) such that, for each $k \geqslant 2$, the subsequence $(c_{kr})_{r=0}^{\infty}$ converges.

4 For $n \geqslant 1$ let k, m be the natural numbers such that $n = 2^{k-1}(2m-1)$ (as in an enumeration of $\mathbb{N}^2$), and define $a_n = \dfrac{k}{m+k}$.

(i) Find a_{312}.
(ii) *Optional* Make a sketch showing the first few (perhaps 2000?) values.
(iii) Show that for each rational number $y \in (0,1)$ there exist infinitely many values of n such that $a_n = y$
(iv) Show that for every real number $x \in [0,1]$ the sequence (a_n) has a subsequence which converges to x.
[*Show first that there is a sequence* (y_r) *of rational numbers in* $(0,1)$ *which converges to* x.]

The following question is ***optional.***

5 (a) Let (a_n) be any sequence which does not converge to 0. Prove that there exist $\varepsilon > 0$ and a subsequence (a_{n_r}) such that $|a_{n_r}| \geqslant \varepsilon$ for all $r \geqslant 1$.
(b) Let (b_n) be a sequence of real numbers and suppose that each subsequence (b_{n_r}) of (b_n) has a subsubsequence $(b_{n_{r_s}})$ which converges to 0. Prove that $b_n \to 0$ as $n \to \infty$.
[*Hint: Argue by contradiction.*]