Analysis I Sheet 5 MT05

Monotonic sequences, Bolzano-Weierstrass, Cauchy sequences

1 In this question we establish in a differnt way the existence of square roots; so you may not use square roots in your solution.

Let $a > 0$. Given any $a_1 > 0$, define a sequence $(a_n)_{n \geqslant 1}$ by $a_{n+1} = \frac{1}{2}\left(a_n + \dfrac{a}{a_n}\right)$ for $n \geqslant 1$.

(i) Prove that $a_n > 0$ for all n, and so the sequence is well-defined.

(ii) Prove that $a_n^2 \geqslant a$ for all $n \geqslant 2$, and hence that $a_{n+1} \leqslant a_n$ for all $n \geqslant 2$.
[*Hint: Express* $a_{n+1}^2 - a$ *as a perfect square.*]

(iii) Prove that (a_n) converges.

(iv) Prove that $\lim_{n\to\infty} a_n > 0$ and $(\lim_{n\to\infty} a_n)^2 = a$.

Deduce that there is unique bijection $x \mapsto \sqrt{x}$ of the positive real numbers satisfying $(\sqrt{x})^2 = x$.

[*Remark: This algorithmic method of finding square roots has been ascribed to Heron of Alexandria but is much older.*]

2 Define sequences (a_n), (b_n) by $a_1 = 1$, $b_1 = 2$ and $a_{n+1} = (a_n b_n)^{1/2}$, $b_{n+1} = \frac{1}{2}(a_n + b_n)$ for $n \geqslant 1$. Prove that

(i) Find the first few terms.

(ii) $a_n < a_{n+1} < b_{n+1} < b_n$ for all $n \geqslant 1$;

(iii) (a_n) and (b_n) both converge;

(iv) the two limits are equal.

[*Remark: You can find all about this sequence by chasing the links after googling the the first few terms.*]

3 Let (z_n) be a bounded sequence of complex numbers. Prove that there is a convergent subsequence (z_{n_r}).

[*Hint: Apply the Bolzano-Weierstrass Theorem to the real and imaginary parts, but be careful!*]

4 Let $a_n = \displaystyle\int_1^n \frac{\cos x}{x^2}\,\mathrm{d}x$. Using the fact that $-1 \leqslant \cos x \leqslant 1$, prove that $|a_m - a_n| \leqslant \dfrac{1}{n}$ for all $m \geqslant n$, and deduce that (a_n) converges.

By integration by parts, or otherwise, show that $\displaystyle\lim_{n\to\infty} \int_1^n \frac{\sin x}{x}\,\mathrm{d}x$ exists.

In this ***optional*** *question we can now establish a 1–1 correspondence between the set of positive real numbers and the set of non-terminating decimals.*

5 Let (t_n) be a sequence of natural numbers, each of which is one of $\{0,1,2,3,4,5,6,7,8,9\}$; let m be a non-negative integer. Suppose that for all n there exists an $N > n$ such that $t_N \neq 0$. Let $d_n := m + \sum_{k=0}^{n} t_k/10^k$. Prove that

(a) (d_n) is a monotone non-decreasing sequence;

(b) (d_n) is a bounded sequence;

(c) (d_n) is convergent with limit t, say;

(d) the n-th decimal truncation of t is d_n.

You can use this correspondence to give a different proof of the uncountability of $\mathbb{R}$.