Analysis I Sheet 6 MT05

Series

1 For which of the following sequences (a_n) is $\sum a_n$ convergent?

$$\text{(i)}\quad \frac{n^4+1}{2^n} \qquad \text{(ii)}\quad \frac{n+2^n}{n2^n} \qquad \text{(iii)}\quad \frac{n!}{n^n}$$

$$\text{(iv)}\quad (1+n^{-1})^{-n} \qquad \text{(v)}\quad n^{-(1+n^{-1})} \qquad \text{(vi)}\quad (-1)^n\frac{\log n}{\sqrt{n}}.$$

[*Hints: Use the Ratio, Comparison and Alternating Series Tests. You may use* $\lim_{n\to\infty}(1+n^{-1})^n = e$ *and* $\lim_{n\to\infty} n^{1/n} = 1$.]

2 Suppose that (a_n) is a sequence of non-negative numbers and that $\sum a_n$ is convergent. Prove that $\sum \dfrac{a_n}{1+a_n}$ is also convergent.

3 Let $n \geqslant 1$. By comparing with a geometric series, prove that $\displaystyle\sum_{r=n+1}^{\infty} \frac{n!}{r!}$ is convergent and its sum is less than $1/n$.

Euler's number e is defined to be $\mathrm{e} = \sum_{r=0}^{\infty} \frac{1}{r!}$. Show that $0 < \mathrm{e} - \sum_{r=0}^{n} \frac{1}{r!} < \frac{1}{n(n!)}$.

[*It now follows from Analysis I Sheet 1, Q.5 that e is irrational*].

4 For each of the following series, determine the values of the real numbers α, β and γ for which the series converges and the values for which it diverges. Justify your answers.

$$\text{(i)}\quad \sum_{n=1}^{\infty}\frac{1}{n^\alpha} \qquad \text{(ii)}\quad \sum_{n=2}^{\infty}\frac{1}{n(\log n)^\beta} \qquad \text{(iii)}\quad \sum_{n=2}^{\infty}\frac{1}{n^\gamma}\left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right).$$

[*Hints: Use the Integral and Comparison Tests. For (iii), show that* $\dfrac{1}{\sqrt{n-1}} - \dfrac{1}{\sqrt{n}} = \dfrac{1}{\sqrt{n}\sqrt{n-1}(\sqrt{n-1}+\sqrt{n})}$*, and so lies between* $\dfrac{1}{2n^{\frac{3}{2}}}$ *and* $\dfrac{1}{2(n-1)^{\frac{3}{2}}}$].

This ***optional*** *question uses the fact proved in lectures, that* $\gamma_n := \sum_{r=1}^{n} \frac{1}{r} - \log n$ *converges to Euler's constant,* γ.

5 By considering the partial sums to $4n$ terms, or otherwise, evaluate the infinite sum

$$1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \dots$$

(the series $\sum(-1)^{r-1}/r$ rearranged to take three positive terms, one negative, three positive, one negative, etc.).

[*Hint: mimic what was done in the lectures to prove that* $\sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} = \log 2$.