Linear Algebra 1: Introduction

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Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Introduction
- Fields and vector spaces
- Subspaces and quotient spaces
- Revision of some Mods linear algebra
- Linear transformations

Welcome to Further Linear Algebra

A reminder of my view:

- You = learner = thinker
- tutorial = primary learning and teaching opportunity
- book = main source of ideas and information
- lectures = oral supplement to books and tutorials

Axioms for Fields

Definition: A *field* is a set F with distinguished elements 0, 1 and with two binary operations $+$ and $\times$ satisfying axioms below (the 'axioms of arithmetic').

Conventionally:

for the image of (a, b) under the function $+ : F \times F \rightarrow F$ we write $a + b$;

for the image of (a, b) under the function $\times : F \times F \rightarrow F$ we write ab;

and $x + yz$ means $x + (yz)$.

The axioms of arithmetic

(1)	$a + (b + c) = (a + b) + c$	[+ is associative]
(2)	$a + b = b + a$	[+ is commutative]
(3)	$a + 0 = a$	
(4)	$(\forall a \in F)(\exists b \in F)(a + b = 0)$	
(5)	$a(bc) = (ab)c$	[× is associative]
(6)	$ab = ba$	[× is commutative]
(7)	$a1 = a$	
(8)	$(\forall a \neq 0)(\exists b \in F)(ab = 1)$	
(9)	$a(b + c) = ab + ac$	[× distributes over +]
(10)	$0 \neq 1$	

Vector spaces

Definition: Let F be a field. For example, F could be $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{F}_p$ (same as $\mathbb{Z}_p$). A vector space over F is a set V with distinguished element 0, with a binary operation $+$, and with a function $\times : F \times V \to V$ satisfying axioms below.

Conventionally:

for the image of (a, b) under the function $+ : V \times V \to V$ we write $a + b$;

for the image of (α, v) under the function $\times : F \times V \to V$ we write αv.

Vector space axioms

(1) $u + (v + w) = (u + v) + w$ [+ is associative]

(2) $u + v = v + u$ [+ is commutative]

(3) $u + 0 = u$

(4) $(\forall u \in V)(\exists v \in V)(u + v = 0)$

(5) $\alpha(\beta v) = (\alpha\beta)v$

(6) $\alpha(u + v) = \alpha u + \alpha v$

(7) $(\alpha + \beta)v = \alpha v + \beta v$

(8) $1v = v$

Note: unquantified axioms are understood to have appropriate quantifiers $\forall u \in V$, ... and/or $\forall \alpha \in F$, ... in front.

Examples

Example 1: F^n is a vector space over F

Example 2: $F[x]$ is a vector space over F

Example 3: $\mathsf{M}_{m \times n}(F)$ is a vector space over F

Example 4: if K is a field and F a subfield then K is a vector space over F.

etc., etc., etc.

Basic theorems

Exactly as for rings, or for vector spaces over $\mathbb{R}$ [recall Mods], one can **prove** important "trivialities". For example:

- $\forall v \in V : 0v = 0;$ $\qquad \forall \alpha \in F : \alpha 0 = 0;$
- $(\forall \alpha \in F)(\forall v \in V) : \alpha v = 0 \Rightarrow \alpha = 0$ or $v = 0;$
- $(\forall \alpha \in F)(\forall v \in V) : \alpha(-v) = -(\alpha v) = (-\alpha)v;$
- etc., etc., etc.

Subspaces

Let F be a field, V a vector space over F.

Definition: A subspace of V is a subset U such that

(1) $0 \in U$,

(2) $u, v \in U \Rightarrow u + v \in U$,

(3) $u \in U,\ \alpha \in F \Rightarrow \alpha u \in U$.

Note: it follows from (3) that $-u \in U$ and so U is an additive subgroup of V.

Note: U is a subspace if and only if $U \neq \emptyset$ and (2) and (3) hold.

Note: We write $U \leqslant V$.

Examples

Example 1: Let $L_1, \ldots, L_m$ be homogeneous linear equations $\sum c_{ij}x_j = 0$ in "variables" $x_1, \ldots, x_n$ with coefficients $c_{ij} \in F$, and let

$$U := \{(x_1, \ldots, x_n) \in F^n \mid L_1, \ \ldots, \ L_m\}.$$

Then $U \leqslant F$

Example 2: Let $F^{[n]}[x] := \{f \in F[x] \mid f = 0 \text{ or } \deg f \leqslant n\}$. Then $F^{[n]}[x] \leqslant F[x]$.

Example 3: Upper triangular matrices form a subspace of $\mathsf{M}_{n\times n}(F)$.

etc., etc., etc.

Quotient spaces

Suppose that $U \leqslant V$ where V is a vector space over a field F. Define the quotient space V/U as follows:

- set $:= \{x + U \mid x \in V\}$ [additive cosets]
- $0 := U$
- addition: $(x + U) + (y + U) := (x + y) + U$
- multiplication by scalars: $\alpha(x + U) := \alpha x + U$

Check: that addition is well defined, multiplication by scalars is well defined, the vector space axioms hold in V/U.

Revision: dimension

- spanning set; linear independence; bases;
- dimension;
- $\dim V = d \;\Rightarrow\; V \cong F^d$;
- basis of a subspace may be extended (usually in many ways) to a basis of V;
- intersection $U \cap W$ of subspaces; sum $U + W$;
- $\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$;
- $\dim V = \dim U + \dim(V/U)$.

Revision: linear transformations

Let F be a field, V_1, V_2 vector spaces over F. A map $T : V_1 \to V_2$ is said to be linear if

$$T0 = 0, \quad T(x + y) = Tx + Ty, \quad \text{and} \quad T(\lambda x) = \lambda(Tx)$$

for all $x, y \in V$ and all $\lambda \in F$.

Note: $T : V_1 \to V_2$ is linear if and only if $T(\alpha x+\beta y) = \alpha Tx+\beta Ty$ for all $x, y \in V$ and all $\alpha, \beta \in F$.

Note: the identity $V \to V$ is linear; if $T : V_1 \to V_2$ and $S : V_2 \to V_3$ are linear then $S \circ T : V_1 \to V_3$ is linear.

Revision: rank and nullity

For a linear transformation $T : V \to W$ define the kernel or null-space by $\mathrm{Ker}\,T := \{x \in V \mid Tx = 0\}$. Then define

$$\mathrm{nullity}\,T := \dim \mathrm{Ker}\,T, \qquad \mathrm{rank}\,T := \dim \mathrm{Im}\,T\,.$$

Rank-nullity Theorem. $\mathrm{nullity}\,T + \mathrm{rank}\,T = \dim V$

Note: the rank-nullity Theorem is a version of the First Isomorphism Theorem, $\mathrm{Im}\,T \cong V/\mathrm{Ker}\,T$.