Linear Algebra 2: Direct sums of vector spaces

Thursday 3 November 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Direct sums of vector spaces
- Projection operators
- Idempotent transformations
- Two theorems
- Direct sums and partitions of the identity

Important note: Throughout this lecture F is a field and V is a vector space over F.

Direct sum decompositions, I

Definition: Let U, W be subspaces of V. Then V is said to be the **direct sum** of U and W, and we write $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{0\}$.

Lemma: *Let U, W be subspaces of V. Then $V = U \oplus W$ if and only if for every $v \in V$ there exist unique vectors $u \in U$ and $w \in W$ such that $v = u + w$.*

Proof.

Projection operators

Suppose that $V = U \oplus W$. Define $P : V \to V$ as follows. For $v \in V$ write $v = u + w$ where $u \in U$ and $w \in W$: then define $P(v) := u$.

Observations:

(1) P is well-defined;

(2) P is linear;

(3) $\operatorname{Im} P = U$, $\operatorname{Ker} P = W$;

(4) $P^2 = P$.

Proofs.

Terminology: P is called the **projection** of V onto U along W.

Notes on projection operators

Note 1. Suppose that V is finite-dimensional. Choose a basis $u_1, \ldots, u_r$ for U and a basis $w_1, \ldots, w_m$ for U. Then the matrix of P with respect to the basis $u_1, \ldots, u_r, w_1, \ldots, w_m$ of V is

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Note 2. If P is the projection onto U along W then $I - P$ is the projection onto W along U.

Idempotent operators: a theorem

Terminology: An operator T such that $T^2 = T$ is said to be **idempotent**.

Theorem. *Every idempotent operator is a projection operator.*

Proof.

A theorem about projections

Theorem. *Let $P : V \to V$ be the projection onto U along W. Let $T : V \to V$ be a linear transformation. Then $PT = TP$ if and only if U and W are T-invariant (that is $TU \leqslant U$ and $TW \leqslant W$).*

Proof.

Direct sum decompositions, II

Definition: V is said to be **direct sum** of subspaces $U_1, \ldots, U_k$, and we write $V = U_1 \oplus \cdots \oplus U_k$, if for every $v \in V$ there exist unique vectors $u_i \in U_i$ for $1 \leqslant i \leqslant k$ such that $v = u_1 + \cdots + u_k$.

Note: $U_1 \oplus U_2 \oplus \cdots \oplus U_k = (\cdots((U_1 \oplus U_2) \oplus U_3) \oplus \cdots \oplus U_k)$.

Note: If $U_i \leqslant V$ for $1 \leqslant i \leqslant k$ then $V = U_1 \oplus \cdots \oplus U_k$ if and only if $V = U_1 + U_2 + \cdots + U_k$ and $U_r \cap \sum_{i \neq r} U_i = \{0\}$ for $1 \leqslant r \leqslant k$.

It is **NOT** sufficient that $U_i \cap U_j = \{0\}$ whenever $i \neq j$.

Note: If $V = U_1 \oplus U_2 \oplus \cdots \oplus U_k$ and B_i is a basis of U_i then $B_1 \cup B_2 \cup \cdots \cup B_k$ is a basis of V. In particular, $\dim V = \sum_{i=1}^{k} \dim U_i$.

Partitions of the identity

Let $P_1, \ldots, P_k$ be linear mappings $V \to V$ such that $P_i^2 = P_i$ for all i and $P_i P_j = 0$ whenever $i \neq j$. If $P_1 + \cdots + P_k = I$ then $\{P_1, \ldots, P_k\}$ is known as a **partition of the identity on V.**

Example: If P is a projection then $\{P, I - P\}$ is a partition of the identity.

Theorem. *Suppose that $V = U_1 \oplus \cdots \oplus U_k$. Let P_i be the projection of V onto U_i along $\bigoplus_{j \neq i} U_j$. Then $\{P_1, \ldots, P_k\}$ is a partition of the identity on V. Conversely, if $\{P_1, \ldots, P_k\}$ is a partition of the identity on V and $U_i := \operatorname{Im} P_i$ then $V = U_1 \oplus \cdots \oplus U_k$.*

Proof.