Linear Algebra 3: Dual spaces

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Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Linear functionals and the dual space
- Dual bases
- Annihilators
- An example
- The second dual.

Important note: **Throughout this lecture F is a field and V is a vector space over F.**

Linear functionals

Definition: a linear functional on V is a function $f : V \to F$ such that

$$f(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 f(v_1) + \alpha_2 f(v_2)$$

for all $\alpha_1, \alpha_2 \in F$ and all $v_1, v_2 \in V$.

Note: thus a linear functional is a linear transformation $V \to F$, where F is construed as a 1-dimensional vector space over itself.

Example: if $V = F^n$ (column vectors) and y is a $1 \times n$ row vector then the map $v \mapsto yv$ is a linear functional on V.

Dual spaces

Definition: The dual space V' of V is defined as follows:

$$\begin{aligned} \text{Set} &:= \text{set of linear functionals on } V \\ 0 &:= \text{zero function} \qquad [v \mapsto 0 \text{ for all } v \in V] \\ (f_1 + f_2)(v) &:= f_1(v) + f_2(v) \qquad [\text{pointwise addition}] \\ (\lambda f)(v) &:= \lambda f(v) \quad [\text{pointwise multiplication by scalars}] \end{aligned}$$

Note: Check that the vector space axioms are satisfied.

Note: Sometimes V' is written V^* or $\mathrm{Hom}(V, F)$ or $\mathrm{Hom}_F(V, F)$.

Dual basis, I

Theorem: *Suppose that V is finite-dimensional. For every basis $v_1, v_2, \ldots, v_n$ of V there is a basis $f_1, f_2, \ldots, f_n$ of V' such that*

$$f_i(v_j) = \begin{cases} 1 & \text{if } i = j\,, \\ 0 & \text{if } i \neq j\,. \end{cases}$$

In particular, $\dim V' = \dim V$.

Proof.

Dual basis, II

Note: The basis $f_1, f_2, \ldots, f_n$ is known as the dual basis of $v_1, v_2, \ldots, v_n$. Clearly, it is unique.

Example: If $V = F^n$ ($n \times 1$ column vectors) then we may identify V' with the space of $1 \times n$ row vectors. The canonical basis $e_1, e_2, \ldots, e_n$ then has dual basis $e_1', e_2', \ldots, e_n'$.

Annihilators

Definition: For a subset U of V the annihilator is defined by

$$U^\circ := \{f \in V' \mid f(u) = 0 \text{ for all } u \in U\}\,.$$

Note: For any subset U the annihilator U° is a subspace. It is $\{f \in V' \mid U \subseteq \operatorname{Ker} f\}$.

Theorem. *Suppose that V is finite-dimensional and U is a subspace. Then*

$$\dim U + \dim U^\circ = \dim V\,.$$

Proof.

A worked example

Part of an old Schools question: Let V be a finite-dimensional vector space over a field F. Show that if U_1, U_2 are subspaces then $(U_1 + U_2)^\circ = U_1^\circ \cap U_2^\circ$ and $(U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ$.

Response.

The second dual

Theorem. *Define* $\Phi : V \to V''$ *by* $(\Phi v)(f) := f(v)$ *for all* $v \in V$ *and all* $f \in V'$. *Then* Φ *is linear and one–one* [*injective*]. *If* V *is finite-dimensional then* Φ *is an isomorphism.*

Proof.