Linear Algebra 4: Dual transformations

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Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Dual transformations
- Matrix of a dual transformation
- Kernel and image of a dual transformation
- A worked example

Important note: **Throughout this lecture F is a field, V and W are vector spaces over F, and $T : V \to W$ is linear.**

Dual transformations

Definition: The dual transformation $T' : W' \to V'$ is defined by $T'f := f \circ T$ for all $f \in W'$.

Note: Thus $(T'f)v = f(Tv)$ for all $v \in V$.

Check that if $f \in W'$ then $T'f \in V'$.

Check that T' is linear.

The matrix of a dual transformation

Theorem: *Suppose that V, W are finite-dimensional over F. Let $v_1, v_2, \ldots, v_m$ be a basis of V, and $w_1, w_2, \ldots, w_n$ a basis of W. Let A be the matrix of T with respect to these bases; let B be the matrix of the dual transformation T' with respect to their dual bases. Then $B = A^{\mathsf{Tr}}$.*

Proof.

Corollary. $\operatorname{rank} T' = \operatorname{rank} T$.

Kernel and image of a dual transformation

Theorem: *Suppose that V, W are finite-dimensional over F. Then*

$$\operatorname{Ker} T' = (\operatorname{Im} T)^\circ \qquad \textit{and} \qquad \operatorname{Im} T' = (\operatorname{Ker} T)^\circ .$$

Proof.

Example: FHS 2000, Paper a1, Qn 1

Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $P : V \to V$ be a linear transformation of V. Let $V_1 = \text{Ker}\,(P)$ and $V_2 = \text{Ker}\,(I_V - P)$, where $I_V : V \to V$ is the identity map on V, and suppose that $V = V_1 \oplus V_2$. Prove that $P^2 = P$.

Define the *dual space* V' of V and the *dual transformation* P' of P. Show that $(P')^2 = P'$. Hence or otherwise show that $V' = U_1 \oplus U_2$ where $U_1 = \text{Ker}\,(P')$ and $U_2 = \text{Ker}\,(I_{V'} - P')$.

Let $\mathcal{E}$ be a basis for V. Define the *dual basis* $\mathcal{E}'$ for V' and show that it is indeed a basis. Suppose that $\mathcal{E} \subseteq V_1 \cup V_2$. Show that $\mathcal{E}' \subseteq U_1 \cup U_2$, and describe the matrices of P and P' with respect to the bases $\mathcal{E}$ and $\mathcal{E}'$ respectively.

[**nb**: We'll do this for vector spaces over an arbitrary field F.]