Linear Algebra 5: A linear transformation on a finite-dimensional vector space

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Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Characteristic polynomial of a linear transformation
- Properties of characteristic polynomial
- Minimal polynomial of a linear transformation
- Properties of minimal polynomial
- Roots of the minimal and characteristic polynomials

Note: Throughout the lecture F is a field, V is a finite-dimensional vector space over F, and $T : V \to V$ is a linear transformation.

Determinants

Recall from Mods: Det A and Trace A of a square matrix A.

Definition: Define the determinant of T by Det T := Det A, where A is the matrix of T with respect to some basis of V. Similarly, define the trace by Trace T := Trace A.

Observation: *Determinant and trace of T do not depend on the basis of V. They depend only on T.*

Proof.

The characteristic polynomial

Definition: The characteristic polynomial of an $n \times n$ matrix A is defined by

$$c_A(x) := \text{Det}\,(xI - A)\,.$$

The characteristic polynomial of T is defined by $C_T(x) := C_A(x)$ where A represents T with respect to some basis of V.

Note: If $n := \dim V$ then $c_T(x)$ is a monic polynomial (monic means leading coefficient $= 1$), and it is of degree n. In fact

$$c_T(x) = x^n - c_1 x^{n-1} + c_2 x^{n-2} - \cdots + (-1)^n c_n\,,$$

where

$$c_1 = \text{Trace}\,T\,, \qquad c_n = \text{Det}\,T, \qquad \textit{etc.}$$

Properties of the characteristic polynomial

- $c_T(x)$ is well defined (independent of basis).

- If $S = U^{-1}TU$, where $U : V \to V$ is linear and invertible, then $c_S(x) = c_T(x)$.

- Roots of $c_T(x)$ are the eigenvalues of T.

- If $\lambda \in F$ is an eigenvalue of T then $\exists v \in V$ such that $v \neq 0$ and $Tv = \lambda v$, that is, there exists an eigenvector for T with eigenvalue λ.

- Conversely, if v is an eigenvector for T, so that $v \neq 0$ and $\exists \lambda \in F$ such that $Tv = \lambda v$, then $c_T(\lambda) = 0$.

The minimal polynomial, I

Lemma. *Let* $n := \dim V$*. The set of all linear transformations* $S : V \to V$ *is a vector space of dimension* n^2*.*

Proof.

Polynomial functions of T: for $f(x) = a_0 + a_1 x + \cdots + a_k x^k \in F[x]$ we define $f(T) := a_0 I + a_1 T + \cdots + a_k T^k$, where $I : V \to V$ is the identity map.

Note: if $f, g \in F[x]$ then $f(T)g(T) = g(T)f(T)$.

The minimal polynomial, II

Lemma. *There is a polynomial* $f \in F[x] \backslash \{0\}$ *such that* $f(T) = 0$*.*
[*Similarly for* $n \times n$ *matrices.*]

Proof.

Definition. A monic polynomial $f \in F[x] \setminus \{0\}$ of least degree such that $f(T) = 0$ is known as the minimal polynomial of T.
[Similarly for $n \times n$ matrices A.]

Examples. $m_I(x) = x - 1$; $m_0(x) = x$; if $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ then $m_A(x) = (x^2 - 3x + 2)$.

Properties of the minimal polynomial, I

- Minimal polynomial is unique.

- We'll write $m_T(x)$ (or $m_A(x)$ when A is an $n \times n$ matrix over F) for the minimal polynomial.

- If $A \in \mathrm{M}_{n\times n}(F)$ and A represents T with respect to some basis of V then $m_T(x) = m_A(x)$.

- If $S = U^{-1}TU$, where $U : V \to V$ is linear and invertible, then $m_S(x) = m_T(x)$.

Properties of the minimal polynomial, II

Theorem. *For* $f \in F[x]$, $f(T) = 0$ *if and only if* $m_T(x)$ *divides* $f(x)$ *in* $F[x]$. [*Similarly for* $A \in \mathsf{M}_{n\times n}(F)$.]

Proof.

Roots of the minimal polynomial

Theorem. *For* $\lambda \in F$, $m_T(\lambda) = 0$ *if and only if* $c_T(\lambda) = 0$. [*Similarly for* $A \in \mathsf{M}_{n\times n}(F)$.]

Proof.

Example. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Here $c_A(x) = (x-1)(x-2)^2$ and $m_A(x) = (x-1)(x-2)$.

Note: in fact $m_T(x)$ and $c_T(x)$ always have the same irreducible factors in $F[x]$.