Linear Algebra 6: The Primary Decomposition Theorem

Friday 11 November 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- The Primary Decomposition Theorem, Mark 1
- The Primary Decomposition Theorem, Mark 2
- The Primary Decomposition Theorem, Mark 3
- An application: diagonalisability

Note: Throughout this lecture F is a field, V is a finite-dimensional vector space over F, and $T : V \to V$ is a linear transformation.

The Primary Decomposition Theorem, Mark 1

Theorem: *Suppose that $f(T) = 0$, where $f \in F[x]$. Suppose also that $f(x) = g(x)h(x)$, where $g, h \in F[x]$ and g, h are co-prime. Then there are T-invariant subspaces U, W of V such that $V = U \oplus W$ and $g(T|_U) = 0$, $h(T|_W) = 0$.*

Proof.

Challenge. Let P be the projection of V onto U along W. Express P as $p(T)$ for some $p \in F[x]$. (Worth a Marsbar.)

The Primary Decomposition Theorem, Mark 2

Theorem. *If* $m_T(x) = g(x)h(x)$ *where* $g, h \in F[x]$ *are monic and co-prime, then* g *is the minimal polynomial of* $T|_U$ *and* h *is the minimal polynomial of* $T|_W$*.*

Proof.

Example. If $m_T(x) = x^2 - x$ then (as we already know) there exist $U, W \leqslant V$ such that $V = U \oplus W$, $T|_U = I_U$ and $T|_W = 0_W$.

The Primary Decomposition Theorem, Mark 3

The Primary Decomposition Theorem. *Suppose that*

$$m_T(x) = f_1(x)^{m_1} f_2(x)^{m_2} \ \ldots \ f_k(x)^{m_k}\,,$$

where $f_1, f_2, \ldots, f_k$ *are distinct monic irreducible polynomials over* F*. Then*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k\,,$$

where $V_1, V_2, \ldots, V_k$ *are* T*-invariant subspaces and the minimal polynomial of* $T|_{V_i}$ *is* $f_i^{m_i}$ *for* $1 \leqslant i \leqslant k$*.*

Proof.

An application: diagonalisability

Definition: The linear transformation T is said to be diagonalisable if there is a basis of V consisting of eigenvectors of T.

Note: Matrix $A \in \mathrm{M}_{n\times n}(F)$ is said to be diagonalisable if there exists an invertible $n \times n$ matrix P over F such that $P^{-1}AP$ is diagonal. And T is diagonalisable if and only if there is a basis of V with respect to which its matrix is diagonal.

Theorem. *Our transformation T is diagonalisable if and only if $m_T(x)$ may be factorised as a product of distinct linear factors in $F[x]$.*

Proof.