Linear Algebra 7: Diagonal and triangular form

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Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Diagonal form revisited
- An example
- Triangular form
- An example

Note: Throughout this lecture F is a field, V is a finite-dimensional vector space over F, $n := \dim V$, and $T : V \to V$ is a linear transformation.

Diagonal form revisited

Theorem. *Our transformation* T *is diagonalisable if and only if* $m_T(x)$ *may be factorised as a product of distinct linear factors in* $F[x]$.

Second Proof of "if".

An Example

Part of FHS 2001, Paper a1, Question 1. State a criterion for the diagonalizability of a linear transformation in terms of its minimum polynomial, and show that if two linear transformations S and T of V are diagonalizable and $ST = TS$, then there is a basis of V with respect to which *both* S and T have diagonal matrices.

Response.

Triangular form

Definition. An $n \times n$ matrix A with entries $a_{ij} \in F$ is said to be upper triangular if $a_{ij} = 0$ when $i > j$.

Note: if A is upper triangular then $c_A(x) = \prod_{i=1}^{n}(x - a_{ii})$, so the eigenvalues of A are its diagonal entries $a_{11}, \ldots, a_{nn}$.

Definition. Our transformation T is said to be triangularisable if there is a basis of V with respect to which its matrix is upper triangular.

Note: the matrix of T with respect to the basis $v_1, \ldots, v_n$ is upper triangular if and only if each subspace $\langle v_1, \ldots, v_r \rangle$ (for $1 \leqslant r \leqslant n$) is T-invariant.

Triangularisability

Theorem. *Our transformation T is triangularisable if and only if $c_T(x)$ may be factorised as a product of linear factors in $F[x]$.*

Proof.

Note: in particular, if $F = \mathbb{C}$ then **every** linear transformation $V \to V$ is triangularisable.

Calculating triangular form

Method: find an eigenvalue λ of T. Then $(T - \lambda I)V$ is a proper T-invariant subspace of V. Find a triangularising basis there, and extend to V.

Example: Find a triangular form of A where

$$A := \begin{pmatrix} 6 & 2 & 3 \\ -3 & -1 & -1 \\ -5 & -2 & -2 \end{pmatrix} .$$