Linear Algebra 8:
The Cayley–Hamilton Theorem

Thursday 17 November 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Marsbar non-presentation ceremony
- The Example from Lecture 7

- The Cayley–Hamilton Theorem
- Important special cases
- Cayley's paper of 1858
- Proofs

The Cayley–Hamilton Theorem

Theorem. *Let V be a finite-dimensional vector space over a field F, and let $T : V \to V$ be a linear transformation. Then $c_T(T) = 0$.*

Equivalently: If A is an $n \times n$ matrix over F then $c_A(A) = 0$.

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Note: We'll work with $n \times n$ matrices over F.

Important special cases

Note 1. Let $A, B \in \mathrm{M}(n \times n, F)$. If $B = P^{-1}AP$, where P is invertible in $\mathrm{M}(n \times n, F)$, then $c_A(A) = 0$ if and only if $c_B(B) = 0$.

Note 2. If A is diagonalisable then $c_A(A) = 0$.

Note 3. If A is triangularisable then $c_A(A) = 0$.

Proofs.

Cayley's paper of 1858

A. Cayley, 'A memoir on the theory of matrices', *Phil. Trans Roy. Soc. London*, 148 (1858), 17–37
= *The Collected Mathematical Papers of Arthur Cayley*, Vol. II, pp. 475–495.

From the introduction: "I obtain the remarkable theorem that any matrix whatever satisfies an algebraical equation of its own order, [...] viz. the determinant, formed out of the matrix diminished by the matrix considered as a single quantity involving the matrix unity, will be equal to zero."

Cayley's paper of 1858 (Continued)

A. Cayley, A memoir on the theory of matrices, 1858.

From §21: "The general theorem before referred to will be best understood by a complete development of a particular case."

Cayley does the case of a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

From §23: "I have verified the theorem, in the next simplest case of a matrix of the order 3, [... ...]; but I have not thought it necessary to undertake the labour of a formal proof of the theorem in the general case of a matrix of any degree."

General proofs of the Cayley–Hamilton Theorem

- Over $\mathbb{C}$ all matrices are triangularisable; we have proved the theorem for triangular matrices.
- If $F \leqslant \mathbb{C}$ then $c_A(A) = 0$.
- For a general proof using adjoint matrices see, for example, T. S. Blyth & E. F. Robertson, *Basic Linear Algebra*, p. 169 or Richard Kaye & Robert Wilson *Linear Algebra*, p. 170.
- For a general proof using 'rational canonical form' see, for example, Peter J. Cameron, *Introduction to Algebra*, p. 154 or Charles W. Curtis, *Linear Algebra*, p. 226.