Linear Algebra 9: Real inner product spaces

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Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Real inner product spaces
- Orthogonality
- Orthonormal sets
- Orthonormal bases
- Orthogonal complements
- The Gram–Schmidt process

Real inner product spaces

Let V be a vector space over $\mathbb{R}$.

Definition: An inner product on V is a function $B : V \times V \to F$ such that for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{R}$,

(1) $B(\alpha u + \beta v, w) = \alpha B(u, w) + \beta B(v, w)$

(2) $B(u, v) = B(v, u)$ [B is symmetric]

(3) $B(u, u) > 0$ [B is positive definite]

Note. Often we find $\langle u, v \rangle$ used for inner products.

Note. In an inner product space we define $||u|| := \langle u, u \rangle^{\frac{1}{2}}$.

Notes on real inner product spaces

Note. From (1) and (2) follows

$$(1') \qquad B(u, \alpha v + \beta w) = \alpha B(u, v) + \beta B(u, w)$$

Note. A function satisfying (1) and (1′) is said to be bilinear.

Example 1: $V = \mathbb{R}^n$ and $B(u, v) = u \cdot v$.

Example 2: $V = C[0, 1]$ (continuous real-valued functions) and $B(f, g) = \int_0^1 f(t)g(t)\,\mathrm{d}t$.

Orthogonality

Let V be a real inner product space.

- for $u \in V$ define $u^\perp := \{v \in V \mid \langle v, u \rangle = 0\}$;
- for $X \subseteq V$ define $X^\perp := \{v \in V \mid \langle v, u \rangle = 0$ for all $u \in X\}$.

Note that $u^\perp$, $X^\perp$ are subspaces of V.

Lemma. *$V = \langle u \rangle \oplus u^\perp$ for any $u \in V$.*

Proof.

Orthogonal and orthonormal sets

Definition. Let V be a real inner product space. Vectors $u_1, u_2, \ldots, u_k$ are said to form an orthogonal set if $\langle u_i, u_j \rangle = 0$ whenever $i \neq j$. They are said to form an orthonormal set if they are orthogonal and $||u_i|| = 1$ for all i.

Lemma: An orthogonal set of non-zero vectors is linearly independent.

Proof.

Orthonormal bases

Theorem. *Let* V *be a finite-dimensional real inner product space, let* $n := \dim V$*, and let* $u \in V \setminus \{0\}$*. There is an orthonormal basis* $v_1, v_2, \ldots, v_n$ *such that* $v_1 = ||u||^{-1}u$*.*

Proof.

Notes on orthonormal bases

Let V be a finite-dimensional real inner product space and let $v_1, \ldots, v_n$ be an orthonormal basis of V. Let $u, w \in V$ and suppose that $u = x_1 v_1 + \cdots + x_n v_n$, $w = y_1 v_1 + \cdots + y_n v_n$, where $x_i, y_j \in \mathbb{R}$ for all i, j.

Note (1): $x_i = \langle u, v_i \rangle$.

Note (2): $||u|| = \left(\sum |x_i|^2 \right)^{\frac{1}{2}}$.

Note (3): $\langle u, w \rangle = \sum x_i y_i$.

Orthogonal complements

Theorem. *Let* V *be a real inner product space. If* U *is a finite-dimensional subspace then* $V = U \oplus U^{\perp}$ *.*

Proof.

The Gram–Schmidt process, I

Theorem. *Let V be a real inner product space and let $u_1, \ldots, u_n$ be linearly independent vectors in V. Then there exists an orthonormal set $v_1, \ldots, v_n$ in V such that*

$$\mathrm{Span}(v_1, \ldots, v_k) = \mathrm{Span}(u_1, \ldots, u_k) \quad \textit{for} \quad 0 \leqslant k \leqslant n\,.$$

Proof.

The Gram–Schmidt process, II

Note 1: The construction in the proof is known as the Gram–Schmidt orthogonalisation process.

Note 2: If T is the transition matrix from $u_1, \ldots, u_n$ to $v_1, \ldots, v_n$ then T is positive upper triangular—that is, upper triangular with positive diagonal entries.