Linear Algebra 11: Inner product spaces, III: Two important inequalities

Thursday 24 November 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Bessel's Inequality
- Some examples
- The Cauchy–Schwarz Inequality
- Some examples

Note: throughout this lecture V is a real or complex inner product space.

Bessel's Inequality

Theorem. *Let* $v_1, \ldots, v_m$ *be an orthonormal set in* V *. If* $u \in V$ *then*

$$\sum_{1}^{m} |\langle u, v_i \rangle|^2 \leqslant ||u||^2 \,.$$

Equality holds if and only if $u \in \mathrm{Span}\{v_1, \ldots, v_m\}$ *.*

Proof.

Some examples

Example: $V = \mathbb{R}^n$ with the usual inner product;

$v_k = e_k$ for $k = 1, 2, \ldots, m$;

$u = (x_1, x_2, \ldots, x_n)^{\text{tr}}$.

Some examples

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$v_k = e_k$ for $k = 1, 2, \ldots, m$;
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Example: V is the space of continuous $f : [0,1] \to \mathbb{C}$;
$\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\,\mathrm{d}t$; $v_k(t) = \mathrm{e}^{2\pi \mathrm{i} kt}$ for $k \in \mathbb{Z}$;
for $f \in V$ define $c_m := \int_0^1 f(t)\mathrm{e}^{2\pi \mathrm{i} mt}\,\mathrm{d}t$;
$\ldots$

The Cauchy–Schwarz Inequality

Theorem. *Let* $u, v \in V$. *Then*

$$|\langle u, v\rangle| \leqslant ||u||.||v||\,.$$

Proof.

Classic alternative proof for $\mathbb{R}$eal inner product spaces.

Challenge: adapt this proof to work over $\mathbb{C}$. [Marsbar for first good solution.]

Examples [FHS 1998, Paper a1, Question 3]

Example: If $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{C}$ then $|\sum a_i b_i|^2 \leqslant \sum |a_i|^2 \sum |b_i|^2$.

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Example: If $a_1, \ldots, a_n \in \mathbb{R}$ and $a_i > 0$ for $1 \leqslant i \leqslant n$ then $(\sum a_i)(\sum 1/a_i) \geqslant n^2$.

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Example: Suppose that $a < b$ and $f, g : [a, b] \to \mathbb{R}$ are continuous. Then $\left(\int_a^b f(x)g(x)\,\mathrm{d}x\right)^2 \leqslant \int_a^b f(x)^2\,\mathrm{d}x \int_a^b g(x)^2\,\mathrm{d}x$.