Linear Algebra 12: Inner product spaces, IV

Friday 25 November 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Isometries of inner product spaces
- A representation theorem for linear functionals
- Adjoints of linear transformations

Note: throughout this lecture V is a real or complex inner product space of finite dimension n.

Isometries

Definition: An isometry of V is a linear transformation $P : V \to V$ such that $\langle Pu, Pv \rangle = \langle u, v \rangle$ for all $u, v \in V$.

Note. An isometry is invertible.

Note. The isometries form a **group:** certainly I is an isometry; if P is an isometry then P^{-1} is an isometry; if P_1, P_2 are isometries then so is $P_2 \circ P_1$.

Orthogonal and unitary transformations

Note. In the $\mathbb{R}$eal case an isometry is known as an orthogonal transformation; the group is the orthogonal group $\mathsf{O}(V)$.

Note. $\mathsf{O}(\mathbb{R}^n) = \{A \in \mathsf{M}_{n\times n}(\mathbb{R}) \mid A^{\mathsf{tr}} = A\}$.

$\mathsf{O}(\mathbb{R}^n)$ is often denoted $\mathsf{O}(n)$ or $\mathsf{O}(n, \mathbb{R})$ or $\mathsf{O}_n(\mathbb{R})$.

Note. In the $\mathbb{C}$omplex case an isometry is known as a unitary transformation; the group is the unitary group $\mathsf{U}(V)$.

Note. $\mathsf{U}(\mathbb{C}^n) = \{A \in \mathsf{M}_{n\times n}(\mathbb{C}) \mid \bar{A}^{\mathsf{tr}} = A\}$.

$\mathsf{U}(\mathbb{C}^n)$ is often denoted $\mathsf{U}(n)$ or $\mathsf{U}(n, \mathbb{C})$ or $\mathsf{U}_n(\mathbb{C})$.

A note about the orthogonal and unitary groups

Observation. O(n) is a closed bounded subset of $M_{n\times n}(\mathbb{R})$.

Similarly, U(n) is a closed bounded subset of $M_{n\times n}(\mathbb{C})$.

A representation theorem

Theorem. *For every* $f \in V'$ *there exists* $v_f \in V$ *such that*

$$f(u) = \langle u, v_f \rangle \quad \textit{for all } u \in V\,.$$

Moreover, v_f *is unique.*

Proof.

Note. This fact is often called the Riesz Representation Lemma.

Note. The map $f \mapsto v_f$ from V' to V is linear in the $\mathbb{R}$eal case, semilinear in the $\mathbb{C}$omplex case.

Adjoints of linear transformations, I

Theorem. *For each linear transformation* $T : V \to V$ *there is a unique linear transformation* $T^* : V \to V$ *such that*

$$\langle Tu, v \rangle = \langle u, T^* v \rangle \quad \textit{for all } u, v \in V\,.$$

Proof.

Adjoints of linear transformations, II

Example. Let $V := \mathbb{R}^n$ with its usual inner product $\langle u, v \rangle = u^{\text{tr}} v$. Suppose $T : V \to V$, $T : v \mapsto Av$ where $A \in \text{M}(n \times n, \mathbb{R})$. Then $T^* : v \mapsto A^{\text{tr}} v$.

Example. Let $V := \mathbb{C}^n$ with its usual inner product $\langle u, v \rangle = u^{\text{tr}} \overline{v}$. Suppose $T : V \to V$, $T : v \mapsto Av$ where $A \in \text{M}(n \times n, \mathbb{C})$. Then $T^* : v \mapsto \bar{A}^{\text{tr}} v$.

Adjoints of linear transformations, III

Theorem. *Let* $S : V \to V$, $T : V \to V$ *be linear transformations. Then:*

$$(S+T)^* = S^*+T^* \,; \quad (\alpha T)^* = \bar{\alpha} T^* \,; \quad (ST)^* = T^* S^* \,; \quad S^{**} = S \,.$$

Proof. Note that $\bar{\alpha}$ is unnecessary, but harmless, in the $\mathbb{R}$eal case.