Linear Algebra 13: Adjoint transformations

Monday 28 November 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Notes on adjoints—from Lecture 12
- The matrix of the adjoint transformation
- Kernel and image of the adjoint transformation
- Self-adjoint linear transformations
- Self-adjoint transformations: eigenvalues and eigenvectors
- Questionnaires

Throughout the lecture V is a finite-dimensional inner product space over $\mathbb{R}$ or $\mathbb{C}$ and $T : V \to V$ is a linear transformation.

The matrix of the adjoint transformation

Theorem. *Let* $\{e_1, e_2, \ldots, e_n\}$ *be an orthonormal basis for* V*. Let* A *be the matrix of* T *with respect to this basis,* A^* *the matrix* T^**. Then*

$$A^* = \bar{A}^{\mathrm{tr}}\,.$$

Proof.

Note: Conjugation is of course not needed if $F = \mathbb{R}$.

Kernel and image of the adjoint transformation

Theorem. $\quad \operatorname{Ker} T^* = (\operatorname{Im} T)^\perp, \quad \operatorname{Im} T^* = (\operatorname{Ker} T)^\perp.$

Proof.

Self-adjoint linear transformations

Note: The transformation T is said to be self-adjoint if $T^* = T$. So T is self-adjoint if and only if $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all $u, v \in V$.

Note: If $S : V \to V$ is any linear transformation then SS^* and S^*S are self-adjoint.

Lemma. *If T is self-adjoint and U is a T-invariant subspace (that is, $TU \subseteq U$) then also $U^\perp$ is T-invariant.*

Proof.

Eigenvalues of a self-adjoint linear transformation

Theorem. *Suppose that T is self-adjoint. Then all eigenvalues of T are $\mathbb{R}$eal—that is, $c_T(x)$ may be factorised as a product of linear factors in $\mathbb{R}[x]$.*

Proof.

Eigenvectors of a self-adjoint linear transformation

Theorem. *Suppose that T is self-adjoint. Let u, v be eigenvectors of T for eigenvalues λ, μ respectively. If $\lambda \neq \mu$ then $\langle u, v\rangle = 0$.*

Proof.

Questionnaires