Rings & Arithmetic 3: Ideals and quotient rings

Friday, 14 October 2005

Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- Ideals, examples
- Quotient rings
- Homomorphisms
- Kernel and image
- The First Isomorphism Theorem
- A worked exercise

Ideals

Definition: A subset A of a ring R (commutative, with 1) is said to be an ideal if

(1) $0 \in A$ and $a, b \in A \Rightarrow a + b, -a \in A$ (so A is an additive subgroup);

(2) $(a \in A, x \in R) \Rightarrow xa \in A$.

Note: If A is an ideal and $1 \in A$ then $A = R$. Thus a proper ideal is never a subring.

Examples of ideals

Examples: $\{0\}$, R are always ideals.

Examples: $n\mathbb{Z}$ is an ideal in $\mathbb{Z}$.

Examples: Generally, if R is any ring (commutative, with 1) and $a \in R$ then aR is an ideal.

Note: Such ideals aR (or Ra) are known as principal ideals. Notations (a) and $\langle a \rangle_R$ are also used by some mathematicians.

Quotient rings

Definition: Let A be an ideal in the ring R. The quotient ring R/A is defined as follows:

$$\begin{aligned} \text{Set} &:= \{x + A \mid x \in R\} \qquad \text{[additive cosets]} \\ 0 &:= A \\ 1 &:= 1 + A \\ (x + A) + (y + A) &:= (x + y) + A \\ (x + A)(y + A) &:= (xy) + A\,. \end{aligned}$$

Check that this is a ring. The issues are:

- are $+$ and $\times$ well-defined?
- do the ring axioms hold?

Important example: $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$

Homomorphisms

Definition: Let R, S be rings (commutative, with 1). A function $\varphi : R \to S$ is said to be a homomorphism if

(0) $\varphi(0) = 0, \quad \varphi(1) = 1$

(1) $\varphi(a+b) = \varphi(a) + \varphi(b)$ for all $a, b \in R$

(2) $\varphi(ab) = \varphi(a)\,\varphi(b)$ for all $a, b \in R$

Example: The identity map $R \to R$ is a homomorphism.

Example: If R is a ring, A an ideal then the map $x \mapsto x + A$ is a homomorphism $R \to R/A$. It is known as the natural projection or natural epimorphism.

Example: In particular, the map $\mathbb{Z} \to \mathbb{Z}_n$ where $x \mapsto \bar{x}$ (and $\bar{x}$ is the residue class of x modulo n) is a surjective homomorphism.

Notes on homomorphisms

Note: If $\varphi : R \to S$ and $\psi : S \to T$ are ring homomorphisms then also $\psi \circ \varphi : R \to T$ is a homomorphism.

Note: If $\varphi : R \to S$ is a ring homomorphism then $\varphi U(R) \leqslant U(S)$.

Definition: An isomorphism is an invertible homomorphism. We write $R \cong S$ to mean that there exists an isomorphism $R \to S$ (and then we say that R, S are isomorphic).

Note: A ring homomorphism $\varphi : R \to S$ is an isomorphism if and only if it is one-one and onto (injective and surjective).

Image and kernel

Definition: Let $\varphi : R \to S$ be a ring homomorphism. We define the image and kernel of φ by

$$\begin{aligned} \mathrm{Im}\varphi &:= \{y \in S \mid \exists x \in R : \varphi(x) = y\} \\ \mathrm{Ker}\varphi &:= \{x \in R \mid \varphi(x) = 0\}\,. \end{aligned}$$

Important Observation: If $\varphi : R \to S$ is a ring homomorphism then $\mathrm{Im}\varphi$ is a subring of S and $\mathrm{Ker}\varphi$ is an ideal in R.

Proof.

The First Isomorphism Theorem

First Isomorphism Theorem for rings: If $\varphi : R \to S$ is a ring homomorphism then $\text{Im}\varphi \cong R/\text{Ker}\varphi$.

Proof.

A worked example

Part of Schools 1987, I, 5. Let D be the ring of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ with the operations of pointwise addition and multiplication. Show that

$$I = \{f \in D : f(0) = f'(0) = 0\}$$

is an ideal in D.

Let $\mathbb{R}[x]$ denote the ring of polynomials in the indeterminate x with real coefficients, and (x^2) the ideal generated by the polynomial x^2. Show that there is a homomorphism from $\mathbb{R}[x]$ onto D/I and deduce that $D/I \cong \mathbb{R}[x]/(x^2)$.