Rings & Arithmetic 4: Some applications of the First Isomorphism Theorem

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Lectures for Part A of Oxford FHS in Mathematics and Joint Schools

- The Second Isomorphism Theorem
- The Third Isomorphism Theorem
- Maximal ideals

The Second Isomorphism Theorem

Theorem: *Let R be a ring (commutative, with 1), S a subring, and A an ideal in R. Then:*

(1) $S + A$ *is a subring of* R;

(2) $S \cap A$ *is an ideal in* S;

(3) $(S + A)/A \cong S/(S \cap A)$.

Proof.

An important example

Example: Let R be a ring (commutative, with 1). Let $f \in R[x]$ with $\deg f \geqslant 1$. Then $R \leqslant R[x]/(f)$.

Note: recall that (f) denotes the principal ideal of $R[x]$ generated by f, that is, the set of all multiples of f.

Note: by $R \leqslant R[x]/(f)$ we mean that there is a "natural" injective homomorphism $R \to R[x]/(f)$.

The Third Isomorphism Theorem

Theorem: *Let R be a ring (commutative, with 1), and A an ideal in R.*

(1) *If B is an ideal in R and $A \subseteq B$ then B/A is an ideal of R/A.*

(2) *The map $B \mapsto B/A$ is a one-one correspondence (that is, a bijection) between ideals of R that contain A and ideals of R/A;*

(3) *If B is an ideal and $A \subseteq B$ then $(R/A)\Big/(B/A) \cong R/B$.*

Proof.

Maximal ideals

Definition: Let R be a ring (commutative, with 1). An ideal A is said to be maximal if $A \neq R$ and the only ideals B such that $A \subseteq B$ are A and R.

Theorem: *Let R be a commutative ring with 1. An ideal A of R is maximal if and only if R/A is a field.*

Proof.

Another important example

Observation: *The ring $\mathbb{Z}_n$ is a field if and only if n is prime.*

Proof.

Observation: *If R is an integral domain of characteristic p, where $p \neq 0$, then $\mathbb{Z}_p \leqslant R$.*

Proof.