Oxford University Department of Mathematics

Part A Further Linear Algebra, MT 2005: Exercise Sheet 1

Revision of Mods linear algebra. Vector spaces over an arbitrary field; subspaces, direct sums, projection operators and their characterisation as idempotent operators. Dual spaces. (See, for example KAYE & WILSON, *Linear Algebra*, Ch. 2, and HALMOS, *Finite-dimensional vector spaces* §§1–13, 15, 17–22, 41,42, or the equivalent in other texts.)

Note: the following problems are offered to give focus to tutorials. The wise and intelligent student will be trying many more exercises, however, from books, past examination papers, and other such sources.

1. (Mods 1974) Let V be a 4-dimensional vector space over a field F (you may assume that F is $\mathbb{R}$ or $\mathbb{C}$), with basis $\{e_1, e_2, e_3, e_4\}$, and let T be the linear transformation defined by

$$\begin{aligned} T(e_1) &= -e_1 - 2e_2 + 2e_3\,, \\ T(e_2) &= 4e_1 + 4e_2 - 5e_3 - 3e_4\,, \\ T(e_3) &= 2e_1 + 2e_2 - 3e_3 - 2e_4\,, \\ T(e_4) &= -e_2 + e_3\,. \end{aligned}$$

Let U be the subspace spanned by $\{e_1 + e_2 - e_3,\ e_1 - e_4,\ -e_1 + e_2 - e_3 + 2e_4\}$. Verify that $T(U) \subseteq U$. Find a basis for U and calculate the matrix of $T|_U$ with respect to this basis. Show that if $F = \mathbb{R}$ then $T|_U$ has no eigenvectors. Find the eigenvalues and eigenvectors if $F = \mathbb{C}$.

2. Let F_n be the n^{th} term of the Fibonacci sequence. Thus $F_0 = 0$, $F_1 = 1$ and the sequence satisfies the recursion $F_n = F_{n-1} + F_{n-2}$. Also, let $\tau := \frac{1}{2}(1+\sqrt{5})$ (so that $\tau : 1$ is the so-called "Golden Ratio"); define $\tau' := \frac{1}{2}(1-\sqrt{5})$ (so that $\tau' = -1/\tau$ and τ, τ' are the roots of the equation $t^2 - t - 1 = 0$); and define $T := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

(i) Let u_n be the column vector $\begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix}$. Show that $T\,u_{n-1} = u_n$.

(ii) Show that τ and τ' are the eigenvalues of T and find corresponding eigenvectors v_τ and $v_{\tau'}$.

(iii) Find real numbers A, B such that $u_1 = Av_\tau + Bv_{\tau'}$.

(iv) Deduce a formula for F_n.

3. Let $U_1, U_2, \ldots$ be *proper* subspaces of a vector space V over a field F (the subspace U is said to be proper if $U \neq V$).

(i) Show that $V \neq U_1 \cup U_2$. [*Hint:* what happens if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$? Otherwise, take $u_1 \in U_1 \setminus U_2$, $u_2 \in U_2 \setminus U_1$, and show that $u_1 + u_2 \notin U_1 \cup U_2$.]

(ii) Show that if $V = U_1 \cup U_2 \cup U_3$ then F must be the field $\mathbb{F}_2$ with just 2 elements. [*Hint:* show first that we cannot have $U_1 \subseteq U_2 \cup U_3$, nor $U_2 \subseteq U_1 \cup U_3$; choose $u_1 \in U_1 \setminus (U_2 \cup U_3)$ and $u_2 \in U_2 \setminus (U_1 \cup U_3)$; observe that if $\lambda \in F \setminus \{0\}$ then $u_1 + \lambda u_2$ must lie in U_3 and exploit this fact.]

(iii) Show that if F is infinite (indeed, if $|F| > n - 1$) then $V \neq U_1 \cup U_2 \cup \cdots \cup U_n$.

p.t.o.

4. Let V be a vector space over $\mathbb{R}$ and let U be a non-trivial proper subspace. Prove that there are infinitely many different subspaces W of V such that $V = U \oplus W$. [*Hint:* think first what happens when V is 2-dimensional; then generalise.]

How far can this be generalised to vector spaces over other fields F?

5. Let V be a vector space (over some field F), and let E_1 and E_2 be projections on V.

(i) Show that $E_1 + E_2$ is a projection if and only if $E_1E_2 + E_2E_1 = 0$. Prove that this happens if and only if $E_1E_2 = E_2E_1$. [*Hint:* consider $E_1E_2E_1$ in two different ways.] Deduce that if $\operatorname{char} F \neq 2$ then $E_1 + E_2$ is a projection if and only if $E_1E_2 = E_2E_1 = 0$.

(ii) Now suppose that $E_1 + E_2$ is a projection. Assuming that $\operatorname{char} F \neq 2$, find its kernel and image in terms of those of E_1 and E_2. What can be said if $\operatorname{char} F = 2$?

6. Let V be a finite dimensional vector space over a field F. Recall that if $X \subseteq V$ then $X^\circ := \{f \in V' \mid f(x) = 0 \text{ for all } x \in X\}$. Prove that if U_1, U_2 are subspaces of V then

$$(U_1 \cap U_2)^\circ = U_1^\circ + U_2^\circ \quad \text{and} \quad (U_1 + U_2)^\circ = U_1^\circ \cap U_2^\circ\,.$$

7. Let F be a field with at least 4 elements and let V be the vector space of polynomials $c_0 + c_1x + c_2x^2 + c_3x^3$ of degree $\leqslant 3$ with coefficients from F.

(i) Show that for $a \in F$ the map $e_a : V \to F$ given by evaluation of polynomial f at a (that is, $e_a(f) = f(a)$) is a linear functional.

(ii) Show that if a_1, a_2, a_3, a_4 are distinct elements of F then $\{e_{a_1}, e_{a_2}, e_{a_3}, e_{a_4}\}$ is a basis of V'.

(iii) Find the basis $\{f_1, f_2, f_3, f_4\}$ of V whose dual basis is the basis in (ii).

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