Oxford University Department of Mathematics

Part A Further Linear Algebra, MT 2005: Exercise Sheet 2

Some theory of a single linear transformation on a finite-dimensional vector space: characteristic polynomial, minimal polynomial, Cayley–Hamilton Theorem, diagonalisability and triangular form. (See, for example, HALMOS, *Finite-dimensional vector spaces*, §§32–40, 46, 47, 53, 54, 56, KAYE AND WILSON, *Linear algebra*, Chs 11, 12, and/or HERSTEIN, *Topics in Algebra*, Second Edition, §6.4.)

Note: the following six problems are offered to give focus to tutorials. The wise and intelligent student will, however, be trying many other exercises from books, past examination papers, and other such sources.

1. For each of the matrices $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 0 & 1 \\ -2 & -1 & -1 & 0 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 1 & -2 \end{pmatrix}$ find the characteristic polynomial and the minimal polynomial.

2. [Part of a former Schools question.] (i) Let A be a 3×3 matrix whose characteristic polynomial is x^3. Show that there are exactly three possibilities for the minimal polynomial of A and give an example of matrices of each type.

(ii) Let V be a finite dimensional vector space and $T : V \to V$ a linear transformation whose minimal polynomial is x^k. Prove that $\{0\} < \operatorname{Ker} T < \operatorname{Ker} T^2 < \cdots < \operatorname{Ker} T^k = V$ and deduce that $\dim V \geqslant k$.

3. Let $T : V \to V$ be a linear transformation of a finite-dimensional vector space over a field F.

(i) Let $m_T(x)$ be its minimal polynomial and let $g(x)$ be a polynomial which is coprime with $m_T(x)$. Show that $g(T)$ is invertible. [Hint: use the fact (from the Rings & Arithmetic course) that there exist $u(x), v(x) \in F[x]$ such that $u(x)m_T(x) + v(x)g(x) = 1$.]

(ii) Using the Primary Decomposition Theorem, or otherwise, deduce that $V = V_1 \oplus V_2$ where V_1, V_2 are T-invariant subspaces (that is $T(V_i) \subseteq V_i$) such that the restriction T_1 of T to V_1 is invertible and the restriction T_2 of T to V_2 is nilpotent (that is, $T_2^m = 0$ for some $m \in \mathbb{N}$).

4. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be a linear transformation. For $\lambda \in F$ define $E_\lambda := \{v \in V \mid Tv = \lambda v\}$.

(i) Check that E_λ is a subspace of V.

(ii) Suppose that $\lambda_1, \ldots, \lambda_n$ are distinct elements of F. For $i = 1, \ldots, n$ let $v_i \in E_{\lambda_i} \setminus \{0\}$. Show that $v_1, \ldots, v_n$ are linearly independent.

(iii) Suppose further that $S : V \to V$ is a linear transformation such that $ST = TS$. Show that $S(E_\lambda) \subseteq E_\lambda$ for each $\lambda \in F$.

5. (i) For each of the matrices X in Qn 1, thought of as matrices with coefficients in $\mathbb{C}$, (a) find an invertible matrix Y over $\mathbb{C}$ such that $Y^{-1}XY$ is diagonal, or prove that no such Y exists, and (b) find an invertible matrix Y over $\mathbb{C}$ such that $Y^{-1}XY$ is upper triangular.

(ii) Can such matrices be found, whose coefficients are real?

6. [A former Schools question.] Prove that an $n \times n$ matrix A with coefficients in a field F is diagonalisable over F if and only if its minimal polynomial splits as a product of distinct linear factors over F.

Let $A := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$. Is A diagonalisable when considered as a matrix over the following fields: (i) $\mathbb{C}$; (ii) $\mathbb{Q}$; (iii) $\mathbb{F}_2$; (iv) $\mathbb{F}_5$?

ΠMN: Queen's: 30.x.2005