Oxford University Department of Mathematics

Part A Further Linear Algebra, MT 2005: Exercise Sheet 3

Real and complex inner product spaces: orthonormal sets and the Gram–Schmidt process; Bessel's inequality; the Cauchy–Schwarz inequality; orthogonal complements. [This is beautifully presented by HALMOS in the first few sections of Chapter III of his book. Alternative sources are HERSTEIN, Ch. 6, §10, COHN, *Algebra*, Vol. 1 (re-published as *Classic algebra*), Ch. 8, and KAYE AND WILSON, *Linear Algebra*, Part II. And several other books in Oxford libraries.]

Note: the following six problems are offered to give focus to tutorials. The wise and intelligent student will be trying many other exercises, however, from books, past examination papers, and other such sources.

1. Express the following quadratic forms as sums or differences of squares of linearly independent linear forms in x, y, z: $x^2 + 2xy + 2y^2 - 2yz - 3z^2$; $xy + yz + xz$. [NB: Probably the easiest method is 'completing the square'.]

2. Let V be a vector space over $\mathbb{R}$, and let $\langle u, v\rangle_1$ and $\langle u, v\rangle_2$ be inner products defined on V. Prove that if $\langle x, x\rangle_1 = \langle x, x\rangle_2$ for all $x \in V$, then $\langle u, v\rangle_1 = \langle u, v\rangle_2$ for all $u, v \in V$.

Does the same result hold for vector spaces over $\mathbb{C}$?

3. [FHS 1992, A1, 3.] For $u := (x, y, z, t) \in \mathbb{R}^4$ we define $q(u) := (x^2 + y^2 + z^2 - t^2)$. A subspace U of $\mathbb{R}^4$ is said to be 'positive' if $q(u) > 0$ for all non-zero u in U, it is said to be 'negative' if $q(u) < 0$ for all non-zero u in U, and it is said to be 'null' if $q(u) = 0$ for all $u \in U$. Prove or disprove each of the following assertions:

(i) if U is positive then $\dim U \leqslant 3$;

(ii) there is a unique positive subspace of dimension 3;

(iii) if U is negative then $\dim U \leqslant 1$;

(iv) there exist non-zero subspaces U, V, W such that U is positive, V is null, W is negative and $\mathbb{R}^4 = U \oplus V \oplus W$.

4. (i) Use the Gram–Schmidt orthogonalisation process to show that for any non-singular $n \times n$ matrix X over $\mathbb{R}$ there exist $n \times n$ matrices P, U over $\mathbb{R}$ such that U is upper triangular with positive entries on its main diagonal, P is orthogonal, and $X = PU$.

(ii) Suppose that $X = QV$, where V is upper triangular with positive entries on its main diagonal, and Q is orthogonal. How must Q be related to P, and V to U?

(iii) What are the corresponding results for non-singular matrices over $\mathbb{C}$?

5. [A former FHS question (corrected).] Let V be a finite-dimensional inner-product space over $\mathbb{R}$. Prove that if $w_1, \ldots, w_m$ is an orthonormal set of vectors in V then

$$||v||^2 \geqslant \sum_1^m (v, w_i)^2 \qquad \text{for all } v \in V. \qquad \text{[Bessel's inequality.]}$$

Now suppose that $u_1, \ldots, u_l$ are unit vectors in V. Prove that $||v||^2 = \sum_1^l (v, u_i)^2$ for all $v \in V$ if and only if $u_1, \ldots, u_l$ is an orthonormal basis for V.

6. Prove that if $z_1, \ldots, z_n \in \mathbb{C}$ then $\Big|\sum_{i=1}^n z_i\Big|^2 \leqslant n \sum_{i=1}^n |z_i|^2$. When does equality hold?

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