Problem sheet (To be done in week 8)

Linear Algebra I, Dr A Henke, MT 2007

Student Number: 654321
Deadline for online homework: Sat Nov 24 18:00:00 2007
This file has been created: Wed Oct 10 16:41:31 2007

The points of this online-homework will be recorded. For every correct answer, you get 1 point; for every wrong answer, you get -1 points. For every question you decide not to answer (indicated by the option -) you get 0 points. Submit your answers via the web-interface.

1	Define the following matrices: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 3 & 5 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $E = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, $G = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Is the given matrix in row reduced echelon form?	
	Matrix E is in row reduced echelon form.	◯ Yes / ◯ No
	Matrix D is in row reduced echelon form.	◯ Yes / ◯ No
	Matrix C is in row reduced echelon form.	◯ Yes / ◯ No
2	Are the following statements about solutions of homogeneous systems of linear equations correct?	
	If u is a solution of a system of linear equations $Ax = 0$, then so is $-2u$.	◯ Yes / ◯ No
	Define $A = \begin{pmatrix} 5 & 0 & \alpha \\ 0 & 2 & 1 \end{pmatrix}$ with $\alpha \in \mathbb{R}$. Then there exists some α such that the system of linear equations $Ax = 0$ has precisely one solution.	◯ Yes / ◯ No
	Let $A \in \mathbb{M}_n(\mathbb{R})$. Assume the row reduced echelon form of A has one zero row. Then the homogeneous system of linear equations $Ax = 0$ has no solution.	◯ Yes / ◯ No
3	Are the following statements about solutions of inhomogeneous systems of linear equations correct?	
	Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$ with $m > n$, and let b be a vector of suitable size with at least one non-zero entry. Then it is possible that the system of linear equations $Ax = b$ has no solution.	◯ Yes / ◯ No
	Define $A = \begin{pmatrix} 3 & -6 \\ 2 & 1 \\ -1 & 2 \end{pmatrix}$. Then there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that the system of linear equations $Ax = (\alpha, \beta, \gamma)^T$ has infinitely many solutions.	◯ Yes / ◯ No
	Define $A = \begin{pmatrix} 5 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$. Then for any non-zero $\alpha, \beta \in \mathbb{R}$, the system of linear equations $Ax = (\alpha, \beta)^T$ has precisely one solution.	◯ Yes / ◯ No
4	Let $$(A\|b) := \left(\begin{array}{ccc\|c} a & 5 & 2 & 230 \\ 2 & 0 & 4 & 120 \\ 1 & 1 & 2 & 70 \end{array}\right)$$ be the matrix corresponding to an inhomogenous system of linear equations $Ax = b$, and assume that $1 \neq a$. Solve the system of linear equations and answer the following questions.	
	If $a = 2$, what is the first component of the solution?	________
	If $a = 4$, what is the first component of the solution?	________
	If $a = 6$, what is the third component of the solution?	________

5	Let $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 2 \\ 3 & 3 & 1 & 3 \end{pmatrix}.$
	Determine $X := B^{-1}$. Enter x_{12}. ____________
	Determine $X := 3 \cdot A^{-1}$. Enter x_{41}. ____________
	Determine $X := 3 \cdot A^{-1}$. Enter x_{11}. ____________
	Determine $X := B^{-1}$. Enter x_{33}. ____________

Please submit your written solutions to the following problems to your college tutor. For this written part of your homework, the deadline set by your college tutor applies.

6 Solve the following system of linear equations using Gaussian elimination:

$$\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & 1 \\ 2x & + & 5y & - & 8z & = & 4 \\ 3x & + & 8y & - & 13z & = & 7 \end{array}.$$

7 Let A be a matrix with entries in $\mathbb{R}$. Prove the following statements:

(a) A system of linear equations $Ax = b$ with fewer equations than variables either has no solution or has several different solutions.

(b) A system of linear equations $Ax = b$ where the rank of A equals the number of equations in the system, always has a solution.

[Denote by B the row reduced echelon form of a given matrix A. Then the rank of A is defined to be the number of non-zero rows of B.]

8 The $n \times n$ *Van der Monde matrix* is the matrix A defined by

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

where $x_1, x_2, \ldots, x_n$ are distinct real numbers. Show that A is invertible. [Hint: let

$$x = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix}$$

be a solution to the simultaneous equations $Ax = 0$. Show that $x_1, \ldots, x_n$ are all roots of the polynomial $a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$.]

9	(Optional.) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that $im(T^2) \subseteq im(T)$ and that $ker(T) \subseteq ker(T^2)$. Prove the equivalence of the following statements: (a) $\mathbb{R}^3 = ker(T) \oplus im(T)$; (b) $ker(T) = ker(T^2)$; (c) $im(T) = im(T^2)$. (We write $\mathbb{R}^3 = ker(T) \oplus im(T)$ if $\mathbb{R}^3 = ker(T) + im(T)$ and $ker(T) \cap im(T) = \{0\}$.)