Problem sheet (To be done in week 9)

Linear Algebra I, Dr A Henke, MT 2007

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There are no online-questions on this last problem sheet. The following problems should be solved during the lecture-free period.

1 (a) Write the following matrix C as a product of elementary matrices:

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

(b) Given are matrices A and B with

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find matrices P and Q such that $PAQ = B$.

2 (a) Determine the row rank and the column rank of the following matrix:

$$X = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 3 & 4 & \dots & n+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n+1 & n+2 & \dots & 2n-1 \end{pmatrix}.$$

(b) Matrix U comes from matrix A by subtracting row one from row three:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(i) Find bases for the two column spaces.

(ii) Find bases for the two row spaces.

(iii) Find bases for the two null spaces.

(c) Let V be a vector space spanned by $\{v_1, \dots, v_n\}$: $V = \text{Span}\{v_1, \dots, v_i, \dots, v_j, \dots v_n\}$. Prove that $V = \text{Span}\{v_1, \dots, v_i + \lambda v_j, \dots, v_n\}$ for any $\lambda \in \mathbb{R}$ and $i \neq j$.

[The null space of a matrix A is by definition the set $\{x \mid Ax = 0\}$.]

3 Let V be a vector space of dimension n and $F : V \to V$ a linear map with $F^2 = F$.

(a) Show that there are subspaces U, W of V with $V = U \oplus W$ and $F(W) = 0$, $F(u) = u$ for all $u \in U$.

(b) Show that there exists a basis B of V and some $r \leq n$ such that

$$M_B^B(F) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

[Here I_r denotes the identity matrix of size r, and 0 denotes zero matrices (of possibly different size).]

4 Define the vectors $a_1 = (1,0,0)$, $a_2 = (0,1,0)$, $a_3 = (0,0,1)$ $a_4 = (2,1,3)$, $b_1 = (1,2,4,1)$, $b_2 = (1,1,0,1)$, $b_3 = (-1,0,4,-1)$ and $b_4 = (0,5,20,0)$.

(i) Show that there is precisely one linear map $f : \mathbb{R}^3 \to \mathbb{R}^4$ with $f(a_i) = b_i$ for $i = 1,2,3,4$.

(ii) Describe the kernel and the image of f and give the rank and the nullity of f.

5 Consider the vector space $\mathbb{R}^4$. Let

$$E := \{(1,-2,6,4),(2,-6,15,8),(0,2,-9,-8),(3,-8,21,7)\}$$

and $S = \{(0,0,1,0),(0,0,0,1)\}$.

(i) Show that E is linearly independent. Why is E a generating set for $\mathbb{R}^4$?

(ii) Use the exchange procedure of Steinitz to get a basis B with $S \subseteq B \subseteq S \cup E$.

6 Let E_2 and E_3 denote the canonical bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively, that is $E_2 = \{(1,0),(0,1)\}$ and $E_3 = \{(1,0,0),(0,1,0),(0,0,1)\}$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$\begin{aligned} f(x,y) &= (x+2y, x-y, 2x+y), \\ g(x,y,z) &= (x-2y+3z, 2y-3z). \end{aligned}$$

(a) Determine the matrices $M_{E_3}^{E_2}(f)$, $M_{E_2}^{E_3}(g)$, $M_{E_2}^{E_2}(g \circ f)$ and $M_{E_3}^{E_3}(f \circ g)$ representing the linear maps $f, g, g \circ f$ and $f \circ g$ with respect to bases E_2 and E_3.

(b) Show that $g \circ f$ is bijective and determine $M_{E_2}^{E_2}((g \circ f)^{-1})$.

7	Consider the vector space W of functions from $\mathbb{R}$ to $\mathbb{R}$. Let $$B = \{sin(x), cos(x), sin(x) \cdot cos(x), sin^2(x), cos^2(x)\},$$ and define $V = \mathrm{Span}(B) \subseteq W$. Consider the map $F : V \to V$ given by $f(x) \mapsto f'(x)$ where f' denotes the first derivative of f. (i) Show that B is a basis of V. (ii) Determine the matrix $M_B^B(F)$. (iii) Give a basis of $\ker(f)$ and $\mathrm{im}(f)$.