The following should be read in conjunction with the article *Finding Rational Points on Bielliptic Genus 2 Curves*, by E.V. Flynn and J.L. Wetherell. It gives the detailed working for Examples 3.2 and 3.3 of that article.

Example 3.2. *Let* $\mathcal{C} : Y^2 = X^6 - 2X^4 - 2X^2 + 1 = (X^2+1)(X^4 - 3X^2 + 1)$. *Then* $\mathcal{C}(\mathbb{Q}) = \{\infty^+, \infty^-, (0, \pm 1), (\pm 2, \pm 5), (\pm 1/2, \pm 5/8)\}$.

Proof. Here $\mathcal{E}^a : Y^2 = F^a(x) = (x+1)(x^2 - 3x + 1)$ and $\mathcal{E}^b : \underline{Y}^2 = F^a(\underline{x}) = (\underline{x}+1)(\underline{x}^2 - 3\underline{x} + 1)$ are the same curve, with torsion group $\{\infty, (-1, 0)\}$. A standard descent shows the $\mathcal{E}^a(\mathbb{Q})$ has rank 1 and

$$\begin{gathered}\{(x_1, Y_1), (x_2, Y_2), (x_3, Y_3), (x_4, Y_4)\} \text{ is a set of representatives for } \mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q}), \\ \text{where } (x_1, Y_1) = \infty, \quad (x_2, Y_2) = (0, 1), \quad (x_3, Y_3) = (4, 5), \quad (x_4, Y_4) = (1/4, 5/8).\end{gathered} \tag{3.12}$$

Similarly, we can take $(\underline{x}_i, \underline{Y}_i) = (x_i, Y_i)$, for $i = 1, \ldots, 4$ to give a set of generators for $\mathcal{E}^b(\mathbb{Q})/2\mathcal{E}^b(\mathbb{Q})$. Let

$$\begin{gathered}\alpha_1 = -1, \quad \alpha_2 = \frac{3}{2} - \frac{1}{2}\sqrt{5}, \quad \alpha_3 = \frac{3}{2} + \frac{1}{2}\sqrt{5}, \\ \text{so that } F^a(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3).\end{gathered} \tag{3.13}$$

Recall, from (1.17), that if $(X, Y) \in \mathcal{C}(\mathbb{Q})$ maps to $(x, Y) = (X^2, Y) = (x_i, Y_i) \in \mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$ and maps to $(\underline{x}, \underline{Y}) = (1/X^2, Y/X^3) = (\underline{x}_j, \underline{Y}_j) \in \mathcal{E}^b(\mathbb{Q})/2\mathcal{E}^b(\mathbb{Q})$, then $(1 - \alpha_k \underline{x}_j)/(x_i - \alpha_k)$ is a square in $\mathbb{Q}(\alpha_k)$ for all of $k = 1, 2, 3$. Let $(m_{i,j})$ be the matrix whose (i, j)th entry is the triple $[(1 - \alpha_1 \underline{x}_j)/(x_i - \alpha_1), (1 - \alpha_2 \underline{x}_j)/(x_i - \alpha_2), (1 - \alpha_3 \underline{x}_j)/(x_i - \alpha_3)]$; note that the first entry of each triple is in $\mathbb{Q}$, whereas the second and third entries are in $\mathbb{Q}(\sqrt{5})$ and are conjugate over $\mathbb{Q}$. On computing $(m_{i,j})$ we find that the entries $m_{1,2} = [1, 1, 1]$, $m_{2,1} = [1, 1, 1]$, $m_{3,4} = [1/4, 1/4, 1/4]$, $m_{4,3} = [4, 4, 4]$ are all triples of squares. This explicitly matches up the work involved in the $\mathcal{E}^a$ approach of Lemma 1.2(a) and the $\mathcal{E}^b$ approach of Lemma 1.2(b), with each $\mathcal{E}^a_{1,k}, \mathcal{E}^a_{2,k}, \mathcal{E}^a_{3,k}, \mathcal{E}^a_{4,k}$ being $\mathbb{Q}(\alpha_k)$-birationally equivalent to $\mathcal{E}^b_{2,k}, \mathcal{E}^b_{1,k}, \mathcal{E}^b_{4,k}, \mathcal{E}^b_{3,k}$, respectively (where the birational transformations preserve the $\mathbb{Q}$-rationality of the x-coordinates). There are no entire rows or columns of non-squares in our 4×4 matrix so, as in Example 3.1, the matrix does not save us from any rank computations, whether $\mathcal{E}^a$ or $\mathcal{E}^b$ is used.

We shall use $\mathcal{E}^a : Y^2 = F^a(x) = (x+1)(x^2-3x+1)$, with the usual map $(X,Y) \mapsto (X^2, Y)$ from $\mathcal{C}$ to $\mathcal{E}^a$. It is sufficient to show that $\infty, (0,\pm 1), (4,\pm 5), (1/4, \pm 5/8)$ give all $(x,Y) \in \mathcal{E}^a(\mathbb{Q})$ for which x is a square in $\mathbb{Q}$. We know that $x = X^2$ and $(x,Y) = (x_1,Y_1), (x_2,Y_2), (x_3,Y_3)$ or (x_4,Y_4) in $\mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$. In the situation where $(x,Y) = (x_1,Y_1) = \infty$ in $\mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$ we know, from (1.15) with $i=1$, that x satisfies both of

$$\begin{aligned} &\mathcal{E}^a_{1,1} : y_1^2 = x(x^2-3x+1), \\ &\mathcal{E}^a_{1,2} : y_2^2 = x(x+1)(x - \frac{3}{2} - \frac{1}{2}\sqrt{5}), \end{aligned} \tag{3.14}$$

for some $y_1 \in \mathbb{Q}$ and $y_2 \in \mathbb{Q}(\sqrt{5})$. Here, and for the rest of this example, we do not bother to write out the third equation $\mathcal{E}^a_{i,3}$ since it is the conjugate of $\mathcal{E}^a_{i,2}$ and gives no new information. A standard descent tells us that $\mathcal{E}^a_{1,1}(\mathbb{Q})$ has rank 0, and $\mathcal{E}^a_{1,1}(\mathbb{Q}) = \{\infty, (0,0)\}$, which corresponds (under $x = X^2$) to already known members of $\mathcal{C}(\mathbb{Q})$. We do not need to use $\mathcal{E}^a_{1,2}$ here (it turns out that $\mathcal{E}^a_{1,2}(\mathbb{Q}(\sqrt{5}))$ also has rank 0).

In the situation where $(x,Y) = (x_2,Y_2) = (0,0)$ in $\mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$ we know, from (1.15) with $i=2$, that x satisfies both of

$$\begin{aligned} &\mathcal{E}^a_{2,1} : y_1^2 = x(x^2-3x+1), \\ &\mathcal{E}^a_{2,2} : y_2^2 = (-\frac{3}{2} + \frac{1}{2}\sqrt{5})x(x+1)(x - \frac{3}{2} - \frac{1}{2}\sqrt{5}), \end{aligned} \tag{3.15}$$

for some $y_1 \in \mathbb{Q}$ and $y_2 \in \mathbb{Q}(\sqrt{5})$. Since $\mathcal{E}^a_{2,1}$ is the same curve as $\mathcal{E}^a_{1,1}$, the rank is again 0, we can apply the same argument as before. This leaves $(x,Y) = (x_3,Y_3)$ or (x_4,Y_4) in $\mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$ as the only potentially non-trivial situations.

In the situation where $(x,Y) = (x_3,Y_3) = (4,5)$ in $\mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$ we know, from (1.15) with $i=3$, that x satisfies both of

$$\begin{aligned} &\mathcal{E}^a_{3,1} : y_1^2 = 5x(x^2-3x+1), \\ &\mathcal{E}^a_{3,2} : y_2^2 = (\frac{5}{2} + \frac{1}{2}\sqrt{5})x(x+1)(x - \frac{3}{2} - \frac{1}{2}\sqrt{5}), \end{aligned} \tag{3.16}$$

for some $y_1 \in \mathbb{Q}$ and $y_2 \in \mathbb{Q}(\sqrt{5})$. Here, $\mathcal{E}^a_{3,1}(\mathbb{Q})$ has rank 1, and so there is no way that $\mathcal{E}^a_{3,1}$ can be used to reduce the possible values of x to a finite set. The rank of $\mathcal{E}^a_{3,2}(\mathbb{Q}(\sqrt{5}))$ is also 1, and so we can apply the Elliptic Curve Chabauty of Section 2, with $\alpha = \sqrt{5}$. It is sufficient to show that $\infty, (0,0), (-1,0), (4,\pm 10)$ are the only $(x,y) \in \mathcal{E}^a_{3,2}(\mathbb{Q}(\sqrt{5}))$

with $\mathbb{Q}$-rational x-coordinate (since then the only $x = X^2$ would correspond to already known (X,Y) in $\mathcal{C}(\mathbb{Q})$). A standard descent shows that $\mathcal{E}^a_{3,2}(\mathbb{Q}(\sqrt{5}))$ is generated by $(0,0), (-1,0), (-1/2+\sqrt{5}/10, -1-\sqrt{5}/5)$, where $(0,0), (-1,0)$ are of order 2 and $P_1 = (-1/2+\sqrt{5}/10, -1-\sqrt{5}/5)$ is of infinite order. Expanding the right hand side of $\mathcal{E}^a_{3,2}$ gives $(5/2+\sqrt{5}/2)x^3 - (5/2+3\sqrt{5}/2)x - (5+2\sqrt{5})x$, so we should take: $g_3 = 5/2+\sqrt{5}/2, g_2 = -5/2-3\sqrt{5}/2, g_1 = -5-2\sqrt{5}, g_0 = 0$ in all of the equations of Section 2. We choose $p = 3$, which satisfies all 6 conditions of (2.13), and compute multiples of $\widetilde{P}_1$ in $\mathbb{F}_3(\sqrt{5})$. We find that the order of $\widetilde{P}_1$ is 3 and so we define as in (2.15),(2,16):

$$m_1 = 3, Q_1 = 3P_1, \text{ where } P_1 = (-1/2+\sqrt{5}/10, -1-\sqrt{5}/5),$$
$$\mathcal{S} = \{\infty, (0,0), (-1,0), (\frac{3}{2}+\frac{\sqrt{5}}{2}, 0), \pm P_1, (0,0)\pm P_1, (-1,0)\pm P_1, (\frac{3}{2}+\frac{\sqrt{5}}{2}, 0)\pm P_1\}, \tag{3.17}$$

so that

$$\text{every } P \in \mathcal{E}^a_{3,2}(\mathbb{Q}(\sqrt{5})) \text{ can be written as } P = S + n_1 Q_1, \text{ for some } S \in \mathcal{S}, n_1 \in \mathbb{Z}. \tag{3.18}$$

Since Q_1 is in the kernel of reduction mod p, $\widetilde{P} = \widetilde{S}$, and so if P has $\mathbb{Q}$-rational x-coordinate then $\widetilde{S}$ must have $\mathbb{F}_p$-rational x-coordinate. Computing the members of $\mathcal{S}$ mod p, we find that this is true only for $S = \infty, (0,0), (-1,0), (4,\pm 10)$, and so these are the only $S \in \mathcal{S}$ we need to consider. We now prove the following five claims.

Claim k. $n_1 = 0$ *is the only* $n_1 \in \mathbb{Z}$ *for which* $R_k + n_1 Q_1$ *has* $\mathbb{Q}$*-rational* x*-coordinate,* where $k = 1, \ldots 5$, and $R_1 = \infty$, $R_2 = (0,0)$, $R_3 = (-1,0)$, $R_4 = (4,10)$, $R_5 = (4,-10)$.

We shall not give all the detailed steps in proving these five claims, since they are similar to those in Example 3.1. It is easily sufficient to work mod 3^8, and we compute the z-coordinate of Q_1 (mod 3^8) as: $3^2(311+166\sqrt{5})$. Substituting this for t in (2.9) gives $\log(z\text{-coordinate of } Q_1)$ (mod 3^8) as: $3^2(14+517\sqrt{5})$. We now substitute $3^2(14+517\sqrt{5})n_1$ for t in (2.10), and see that

$$\begin{aligned} &z\text{-coordinate of } n_1Q_1 = \exp(n_1 \log(z\text{-coordinate of } Q_1)) = O(n_1) \in \mathbb{Z}_3[\sqrt{5}][[n_1]] \\ &z\text{-coordinate of } n_1Q_1 \equiv 3^2(14+517\sqrt{5})n_1 + 3^5(11+14\sqrt{5})n_1^3 \pmod{3^8}. \end{aligned} \tag{3.19}$$

Just as in Example 3.1, we replace z by (3.19) in (2.6) to get $\phi_{R_1}(n_1) = \phi_\infty(n_1)$. For $k = 2, \ldots 5$, replace z by (3.19) and (x_0, y_0) by R_k in (2.8) to get $\phi_{R_k}(n_1)$. Then, for each k, write $\phi_{R_k}(n_1) = \phi^{(0)}_{R_k}(n_1) + \phi^{(1)}_{R_k}(n_1)\sqrt{5}$, where each of $\phi^{(0)}_{R_k}, \phi^{(1)}_{R_k}$ is in $\mathbb{Z}_3[[n_1]]$. The resulting power series $\phi^{(1)}_{R_k}$ are as follows.

$$\begin{aligned}
&\phi^{(1)}_{R_k}(n_1) = O(n_1^2) \in \mathbb{Z}_3[[n_1]] \text{ for k=1,2,3,} \quad \phi^{(1)}_{R_k}(n_1) = O(n_1) \in \mathbb{Z}_3[[n_1]] \text{ for k=4,5.}\\
&\phi^{(1)}_{R_1}(n_1) \equiv 3^4 \cdot 13 \cdot n_1^2 + 3^7 \cdot 2 \cdot n_1^4 \pmod{3^8}\\
&\phi^{(1)}_{R_2}(n_1) \equiv 3^4 \cdot 76 \cdot n_1^2 + 3^7 \cdot n_1^4 \pmod{3^8}\\
&\phi^{(1)}_{R_3}(n_1) \equiv 3^6 \cdot 2 \cdot n_1^2 + 3^7 \cdot n_1^4 \pmod{3^8}\\
&\phi^{(1)}_{R_4}(n_1) \equiv 3^2 \cdot 134 \cdot n_1 + 3^4 \cdot 35 \cdot n_1^2 + 3^5 \cdot 16 \cdot n_1^3 + 3^7 \cdot n_1^4 \pmod{3^8}\\
&\phi^{(1)}_{R_5}(n_1) = \phi^{(1)}_{R_4}(-n_1) \equiv 3^2 \cdot 595 \cdot n_1 + 3^4 \cdot 35 \cdot n_1^2 + 3^5 \cdot 11 \cdot n_1^3 + 3^7 \cdot n_1^4 \pmod{3^8}
\end{aligned} \tag{3.20}$$

As usual, for each k, if $R_k + n_1P_1$ has $\mathbb{Q}$-rational x-coordinate then $\phi^{(1)}_{R_k}(n_1) = 0$. On applying Strassman's Theorem, it is clear that $n_1 = 0$ is the only solution in each case, which proves all five claims.

In the final situation where $(x, Y) = (x_4, Y_4) = (1/4, 5/8)$ in $\mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$ we know, from (1.15) with $i = 4$, that x satisfies both of

$$\begin{aligned}
&\mathcal{E}^a_{4,1} : y_1^2 = \frac{5}{4}x(x^2 - 3x + 1),\\
&\mathcal{E}^a_{4,2} : y_2^2 = (-\frac{5}{4} + \frac{1}{2}\sqrt{5})x(x+1)(x - \frac{3}{2} - \frac{1}{2}\sqrt{5}),
\end{aligned} \tag{3.21}$$

for some $y_1 \in \mathbb{Q}$ and $y_2 \in \mathbb{Q}(\sqrt{5})$. Here again, $\mathcal{E}^a_{4,1}(\mathbb{Q})$ has rank 1, and so there is no way that $\mathcal{E}^a_{4,1}$ can be used to reduce the possible values of x to a finite set. The rank of $\mathcal{E}^a_{4,2}(\mathbb{Q}(\sqrt{5}))$ is also 1, and so we can again apply the Elliptic Curve Chabauty of Section 2, with $\alpha = \sqrt{5}$. A virtually identical argument to that used for $\mathcal{E}^a_{3,2}$ shows that $\infty, (0,0), (-1,0), (1/4, \pm 5/16)$ are the only $(x, y) \in \mathcal{E}^a_{4,2}(\mathbb{Q}(\sqrt{5}))$ with $\mathbb{Q}$-rational x-coordinate.

In summary, for any $(X, Y) \in \mathcal{C}(\mathbb{Q})$, we must have $x = X^2$ satisfy at least one of the four situations described by (3.14),(3.15),(3.16),(3.21). In each of of these situations, we have shown that the only possible $x \in \mathbb{Q}$ are $x = \infty, 0, 4, , 1/4, -1$. Since $x = X^2$ is a square, we can discard $x = -1$, and we are left with $\infty, 0, \pm 2, \pm 1/2$ as the only possibilities for X, as required. $\square$

For $\mathcal{C} : Y^2 = F(X^2) = f_3X^6 + f_2X^4 + f_1X^2 + f_0$, we have now seen an example (Example 3.1) where $F(x)$ is irreducible over $\mathbb{Q}$, and an example (Example 3.2) where

$F(x)$ factors over $\mathbb{Q}$ as a linear and a quadratic. The only remaining situation is when $F(x) = f_3(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$, where all of $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}$. In this case, all of the elliptic curves $\mathcal{E}^a_{i,k}$ in (1.15) are defined over $\mathbb{Q}$, and so the Elliptic Curve Chabauty of Section 2 is not needed. The method will work if, for each i, at least one of $\mathcal{E}^a_{i,1}, \mathcal{E}^a_{i,2}, \mathcal{E}^a_{i,3}$ has rank 0 over $\mathbb{Q}$. We conclude by sketching a quick example of this.

Example 3.3. *Let $\mathcal{C} : Y^2 = (X^2+1)(X^2-3)(X^2+7)$. Then $\mathcal{C}$ has no $\mathbb{Q}$-rational affine points; that is, $\mathcal{C}(\mathbb{Q}) = \{\infty^+, \infty^-\}$.*

Proof. Here, $\mathcal{E}^a : Y^2 = (x+1)(x-3)(x+7)$ and $\mathcal{E}^b : \underline{Y}^2 = (\underline{x}+1)(-3\underline{x}+1)(7\underline{x}+1)$. A standard descent shows that $\mathcal{E}^a(\mathbb{Q})$ has torsion group $\{\infty, (-1,0), (3,0), (-7,0)\}$ and rank 1, with generator $(-5,8)$ of infinite order. We can take $(x_1, Y_1) = \infty$, $(x_2, Y_2) = (-1,0)$, $(x_3, Y_3) = (3,0)$, $(x_4, Y_4) = (-7,0)$, $(x_5, Y_5) = (-5,8)$, $(x_6, Y_6) = (5,12)$, $(x_7, Y_7) = (-2,-5)$, $(x_8, Y_8) = (23,-120)$, so that $\{(x_1, Y_1), \ldots, (x_8, Y_8)\}$ is a set of representatives for $\mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$. Similarly, $\mathcal{E}^b(\mathbb{Q})$ has torsion group $\{\infty, (-1,0), (1/3,0), (-1/7,0)\}$ and rank 1, with generator $(0,1)$ of infinite order. We can take $(\underline{x}_1, \underline{Y}_1) = \infty$, $(\underline{x}_2, \underline{Y}_2) = (-1,0)$, $(\underline{x}_3, \underline{Y}_3) = (1/3,0)$, $(\underline{x}_4, \underline{Y}_4) = (-1/7,0)$, $(\underline{x}_5, \underline{Y}_5) = (0,1)$, $(\underline{x}_6, \underline{Y}_6) = (1/7, 8/7)$, $(\underline{x}_7, \underline{Y}_7) = (-11/7, 40/7)$, $(\underline{x}_8, \underline{Y}_8) = (-3,20)$, so that $\{(\underline{x}_1, \underline{Y}_1), \ldots, (\underline{x}_8, \underline{Y}_8)\}$ is a set of representatives for $\mathcal{E}^b(\mathbb{Q})/2\mathcal{E}^b(\mathbb{Q})$. Let $\alpha_1 = -1, \alpha_2 = 3, \alpha_3 = -7$, so that $F^a(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$. Recall, from (1.17), that if $(X,Y) \in \mathcal{C}(\mathbb{Q})$ maps to $(x,Y) = (X^2, Y) = (x_i, Y_i) \in \mathcal{E}^a(\mathbb{Q})/2\mathcal{E}^a(\mathbb{Q})$ and maps to $(\underline{x}, \underline{Y}) = (1/X^2, Y/X^3) = (\underline{x}_j, \underline{Y}_j) \in \mathcal{E}^b(\mathbb{Q})/2\mathcal{E}^b(\mathbb{Q})$, then $(1-\alpha_k \underline{x}_j)/(x_i - \alpha_k)$ is a square in $\mathbb{Q}$ (each $\alpha_k \in \mathbb{Q}$, so $\mathbb{Q}(\alpha_k) = \mathbb{Q}$ here) for all of $k = 1,2,3$. Let $(m_{i,j})$ be the matrix whose (i,j)th entry is the triple $[(1-\alpha_1\underline{x}_j)/(x_i-\alpha_1), (1-\alpha_2\underline{x}_j)/(x_i-\alpha_2), (1-\alpha_3\underline{x}_j)/(x_i-\alpha_3)]$; note that the third member of each triple is guaranteed to be the product modulo squares of the first two members. Since we are only interested in whether or not each triple contains all squares, we write out $(m_{i,j})$, but with each triple abbreviated to the first two members $[(1-\alpha_1\underline{x}_j)/(x_i-\alpha_1), (1-\alpha_2\underline{x}_j)/(x_i-\alpha_2)]$. All square entries are in boldface.

$$\begin{pmatrix}
[-21,63] & [-24,\mathbf{4}] & [\frac{4}{3},40] & [\frac{6}{7},\frac{10}{7}] & [\mathbf{1},\mathbf{1}] & [\frac{8}{7},\frac{4}{7}] & [-\frac{4}{7},\frac{40}{7}] & [-2,10] \\
[\frac{7}{8},-\frac{63}{4}] & [\mathbf{1},-1] & [-\frac{1}{18},-10] & [-\frac{1}{28},-\frac{5}{14}] & [-\frac{1}{24},-\frac{1}{4}] & [-\frac{1}{21},-\frac{1}{7}] & [\frac{1}{42},-\frac{10}{7}] & [\frac{1}{12},-\frac{5}{2}] \\
[-\frac{21}{4},\frac{63}{40}] & [-6,\frac{1}{10}] & [\frac{1}{3},\mathbf{1}] & [\frac{3}{14},\frac{1}{28}] & [\frac{1}{4},\frac{1}{40}] & [\frac{2}{7},\frac{1}{70}] & [-\frac{1}{7},\frac{1}{7}] & [-\frac{1}{2},\frac{1}{4}] \\
[\frac{7}{2},-\frac{63}{10}] & [\mathbf{4},-\frac{2}{5}] & [-\frac{2}{9},-4] & [-\frac{1}{7},-\frac{1}{7}] & [-\frac{1}{6},-\frac{1}{10}] & [-\frac{4}{21},-\frac{2}{35}] & [\frac{2}{21},-\frac{4}{7}] & [\frac{1}{3},-1] \\
[\frac{21}{4},-\frac{63}{8}] & [6,-\frac{1}{2}] & [-\frac{1}{3},-5] & [-\frac{3}{14},-\frac{5}{28}] & [-\frac{1}{4},-\frac{1}{8}] & [-\frac{2}{7},-\frac{1}{14}] & [\frac{1}{7},-\frac{5}{7}] & [\frac{1}{2},-\frac{5}{4}] \\
[-\frac{7}{2},\frac{63}{2}] & [-4,2] & [\frac{2}{9},20] & [\frac{1}{7},\frac{5}{7}] & [\frac{1}{6},\frac{1}{2}] & [\frac{4}{21},\frac{2}{7}] & [-\frac{2}{21},\frac{20}{7}] & [-\frac{1}{3},5] \\
[21,-\frac{63}{5}] & [24,-\frac{4}{5}] & [-\frac{4}{3},-8] & [-\frac{6}{7},-\frac{2}{7}] & [-1,-\frac{1}{5}] & [-\frac{8}{7},-\frac{4}{35}] & [\frac{4}{7},-\frac{8}{7}] & [2,-2] \\
[-\frac{7}{8},\frac{63}{20}] & [-1,\frac{1}{5}] & [\frac{1}{18},2] & [\frac{1}{28},\frac{1}{14}] & [\frac{1}{24},\frac{1}{20}] & [\frac{1}{21},\frac{1}{35}] & [-\frac{1}{42},\frac{2}{7}] & [-\frac{1}{12},\frac{1}{2}]
\end{pmatrix}. \tag{3.22}$$

In contrast to our previous examples, we gain a significant benefit from computing this matrix. Only the entry $m_{1,5}$ is a triple of squares. This means that in the $\mathcal{E}^a$ approach of Lemma 1.2(a), we need only consider $\mathcal{E}^a_{1,k}$, for $k = 1, 2, 3$, and can ignore $\mathcal{E}^a_{i,k}$ for $i = 2, \ldots, 8$. Similarly, in the $\mathcal{E}^b$ approach of Lemma 1.2(b), we need only consider $\mathcal{E}^b_{5,k}$ ($\mathbb{Q}$-birationally equivalent to $\mathcal{E}^a_{1,k}$), and can ignore all other $\mathcal{E}^b_{j,k}$.

As usual, we use $\mathcal{E}^a : Y^2 = (x+1)(x-3)(x+7)$, with map $(X, Y) \mapsto (X^2, Y)$ from $\mathcal{C}$ to $\mathcal{E}^a$. We have already established that we need only consider $\mathcal{E}^a_{1,k}$, for $k = 1, 2, 3$. Using (1.15), we see that $\mathcal{E}^a_{1,1}$ is the curve $y_1^2 = x(x-3)(x+7)$, and $\mathcal{E}^a_{1,1}(\mathbb{Q})$ has rank 0 with torsion group $\{\infty, (0,0), (3,0), (-7,0)\}$. We do not even need to compute the ranks of $\mathcal{E}^a_{1,2}, \mathcal{E}^a_{1,3}$, since if $(X, Y) \in \mathcal{C}$ and $x = X^2$, we know that x must be an x-coordinate on all three of $\mathcal{E}^a_{1,k}$. This tells us that the only eligible $x \in \mathbb{Q}$ are $x = \infty, 0, 3, -7$. Of these, only $\infty, 0$ are squares, so that the only possible X-coordinates are ∞ and 0. But substituting $X = 0$ into $\mathcal{C}$ gives $Y^2 = -21$ which does not correspond to a point in $\mathcal{C}(\mathbb{Q})$, hence we are left only with $\mathcal{C}(\mathbb{Q}) = \{\infty^+, \infty^-\}$, as required.

Even though it is not required here, if we now gratuitously compute by standard descents the ranks of the 24 elliptic curves $\mathcal{E}^a_{i,k}$ in (1.15), the results are given by the following matrix, whose (k, i) entry is the rank of $\mathcal{E}^a_{i,k}(\mathbb{Q})$.

$$\begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 2 & 1 & 2 \\
1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0
\end{pmatrix} \tag{3.23}$$

This gives us an alternative proof which makes no use of the matrix (3.22), as follows. We know that if $(X,Y) \in \mathcal{C}(\mathbb{Q})$ then (for some $1 \leqslant i \leqslant 8$) $x = X^2$ is the x-coordinate of a point on each of $\mathcal{E}^a_{i,1}(\mathbb{Q}), \mathcal{E}^a_{i,2}(\mathbb{Q}), \mathcal{E}^a_{i,3}(\mathbb{Q})$. The fact that each column of the above matrix contains at least one zero, together with the fact that the torsion group of each $\mathcal{E}^a_{i,k}(\mathbb{Q})$ is just the 2-torsion group, tells us that the only eligible $x \in \mathbb{Q}$ are $x = \infty, 0, -1, 3, -7$. The fact that $\mathcal{C}(\mathbb{Q}) = \{\infty^+, \infty^-\}$ now follows as before. $\square$

In the above example, it was clearly better to have computed the matrix (3.22), since it cheaply avoids all of the rank computations of (3.23) from column 2 onwards.