Honour Moderations: Linear Algebra
Problem Sheet 1

Hilary Term 2005

1. Determine how the rank of the real matrix

$$\begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & b & -1 \end{pmatrix}$$

depends on the real number b.

For which values of a, b and c are the equations

$$\begin{array}{rcrcrcr} 3x & + & y & + & 2z & = & -1 \\ x & + & 2y & - & z & = & a \\ x & & & + & z & = & -1 \\ 2x & + & by & - & z & = & c \end{array}$$

consistent?

Find the most general solution over the real field in all cases of consistency.
[Mods 2002]

2. Let U_1 and U_2 be subspaces of a finite-dimensional vector space V over $\mathbb{R}$. State and prove a formula relating the dimensions of U_1, U_2, $U_1 + U_2$ and $U_1 \cap U_2$.

Let $T : V \to V$ be a linear transformation, and let $I : V \to V$ be the identity transformation.

(i) By considering the equation $v = Tv + (Iv - Tv)$ or otherwise, prove that

$$\operatorname{im} T + \operatorname{im}(I - T) = V.$$

(ii) Show that if $u, v \in V$ satisfy

$$Tu = v - Tv$$

then $v \in \operatorname{im} T$ and $v - Tv \in \operatorname{im}(T - T^2)$. Hence show that

$$\operatorname{im} T \cap \operatorname{im}(I - T) = \operatorname{im}(T - T^2).$$

(iii) Deduce that

$$r_T + r_{I-T} - r_{T-T^2} = \dim V$$

where, if S is any linear transformation, then r_S denotes the rank of S.

[Mods 1993]

3. Let V be an n-dimensional real vector space with basis $v_1, \ldots, v_n$ and let T be the linear transformation on V whose matrix with respect to the above basis is the $n \times n$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Prove that $\ker T = \langle v_1 \rangle$.

Now let U be a non-zero subspace such that $T(U) \subseteq U$. Prove the following statements:

(i) $v_1 \in U$ [**Hint:** consider $T^r(u)$ for $u \in U$ and suitable r];

(ii) $\dim T(U) = \dim U - 1$ [**Hint:** consider the restriction of T to U];

(iii) If U' is a non-zero subspace such that $T(U') \subseteq U'$ and $T(U') = T(U)$ then $U = U'$.

Hence show that $U = \langle v_1, \ldots v_i \rangle$ for some i, $1 \le i \le n$. [Mods 1997]

4. Consider the system

$$\begin{array}{rcrcrcr} \alpha x & + & 2y & + & 3z & = & 0 \\ x & + & \beta y & + & 3z & = & 0 \\ x & + & 2y & + & \gamma z & = & 0. \end{array}$$

Find necessary and sufficient conditions on α, β and γ for the space of solutions to be

(i) 0-dimensional;

(ii) 2-dimensional.

[Mods 1991 (part of question)]

5. Let $S : U \to V$ and $T : V \to W$ be linear transformations of the finite-dimensional vector spaces U, V and W over $\mathbb{R}$.

Show that $r(TS) \le \min(r(S), r(T))$ and $n(TS) \le n(S) + n(T)$, where $r(S)$ and $n(S)$ denote respectively the rank and nullity of S.

In the case $U = V = W$ deduce that $n(TS) \le 2n(ST)$.

[**Hint:** you may wish to apply the rank-nullity formula to the restriction of T to $\operatorname{im} S$.] [Mods 1986]

G.A.S.