Honour Moderations: Linear Algebra
Problem Sheet 2

Hilary Term 2005

1. In each of the following cases, find the matrix A which represents the linear mapping $T : V \to W$ with respect to the initial basis $\mathcal{E}$ and final basis $\mathcal{F}$.

(i) $V = \mathbb{R}^3$, $W = \mathbb{R}^2$, $\mathcal{E}$ and $\mathcal{F}$ are the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, and

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x - y \\ x + 2y - z \end{pmatrix}.$$

(ii) V is the vector space spanned by $\mathcal{E} = \{e^x, xe^x, x^2e^x, x^3e^x\}$ (a subspace of the vector space of all functions $\mathbb{R} \to \mathbb{R}$), $W = V$, $\mathcal{F} = \mathcal{E}$, and $Tf(x) = f'(x)$, the derivative of $f(x)$.

(iii) $V = M_{2\times 2}(\mathbb{R}) = W$, $\mathcal{E} = \mathcal{F} = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, where E_{ij} is the 2×2 real matrix with 1 in the (i, j) position and 0s elsewhere, and

$$TB = CB \text{ where } C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

2. Let $\mathcal{E}$ be the standard basis for $\mathbb{R}^3$, and let $\mathcal{F} = \{f_1, f_2, f_3\}$ where

$$f_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; f_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, f_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y + z \\ x - 4y \\ 3x \end{pmatrix}.$$

Find the matrix of T with respect to (i) the basis $\mathcal{E}$, (ii) initial basis $\mathcal{E}$ and final basis $\mathcal{F}$, and (iii) the basis $\mathcal{F}$.

3. The matrix of a linear mapping $T : V \to V$ with respect to a basis $\{e_1, e_2, e_3\}$ is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

Find its matrix with respect to $\{f_1, f_2, f_3\}$ where

$$f_1 = e_2 - e_3, \; f_2 = e_1 - e_2 + e_3, \; f_3 = -e_1 + e_2.$$

4.

(a) Let V and W be finite-dimensional vector spaces over $\mathbb{R}$. Let $T : V \to W$ be a linear transformation, let $\mathcal{X}$ and $\mathcal{Y}$ be bases of V and W respectively, and let A be the matrix of T with respect to $\mathcal{X}$ and $\mathcal{Y}$. Suppose B is the matrix of T with respect to new bases $\mathcal{X}'$ and $\mathcal{Y}'$ of V and W respectively. Show how A and B are related.

(b) Let V be the vector space over $\mathbb{R}$ of all twice-differentiable real-valued functions f of $x \in \mathbb{R}$ satisfying $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = f$. Let $W = V$, and T be the linear transformation $T(f) = \frac{\mathrm{d}f}{\mathrm{d}x}$. Let $\mathcal{X} = \mathcal{Y} = \{\sinh x, \cosh x\}$, and $\mathcal{X}' = \mathcal{Y}' = \{e^x, e^{-x}\}$. Write down the matrices of T with respect to $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}'$, $\mathcal{Y}'$, and verify that they are related in the way you have claimed in (a). [Mods 1994 (part of question)]

5. Let V be a vector space with basis $e_1, \ldots, e_m$ over $\mathbb{R}$, and W a vector space with basis $f_1, \ldots, f_n$ over $\mathbb{R}$. A linear mapping $T : V \to W$ may be described with respect to the given bases by the $n \times m$ matrix M where $T(e_j) = \sum_i m_{ij} f_i$. Let $e'_1, \ldots, e'_m$ and $f'_1, \ldots, f'_n$ be new bases, given by $e'_k = \sum_i a_{ik} e_i$, $f'_l = \sum_j b_{jl} f_j$. In the case $m = n = 3$ consider the mapping T described by the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

with respect to the given bases. Find new bases with respect to which T is described by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

[Mods 1990 (part of question)]

G.A.S.