Honour Moderations: Linear Algebra
Problem Sheet 4

Hilary Term 2005

1. Let A be each of the following matrices in turn:

$$\begin{pmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ -1 & 4 & -1 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Find all the eigenvectors of A; determine whether A is diagonalisable (over $\mathbb{R}$) and, if so, find an invertible real matrix X for which $X^{-1}AX$ is diagonal.

2. Let T be a linear transformation $T : V \to V$, where V is a finite-dimensional vector space over $\mathbb{R}$.

(i) Show that if T is invertible then the eigenvalues of T^{-1} are the reciprocals of the eigenvalues of T. What can be said about the corresponding eigenvectors?

(ii) Show that if v is an eigenvector of T associated with the eigenvalue λ and k is any natural number, then v is an eigenvector of T^k associated with the eigenvalue λ^k.

3. For any polynomial $p(x) = a_0 + a_1 x + \ldots + a_k x^k$ and any square matrix A, the matrix $p(A)$ is defined as $p(A) = a_0 I + a_1 A + \ldots + a_k A^k$. Show that if v is any eigenvector of A and $c(x)$ is the characteristic polynomial of A, then $c(A)v = 0$. Deduce that if A is diagonalisable then $c(A)$ is the zero matrix.

4. Let $M = \begin{pmatrix} -5 & 3 \\ 6 & -2 \end{pmatrix}$.

(i) Find a diagonal matrix D and an invertible matrix X such that $M = XDX^{-1}$.

(ii) Find at least one cube root of M, by observing that if $D = E^3$ then $M = (XEX^{-1})^3$.

(iii) Express the infinite series $e^M = \sum_{n=0}^{\infty} \frac{1}{n!} M^n$ (where $M^0 = I$) as a 2×2 matrix with entries involving the constant e. [You may assume any general properties of infinite series of matrices that you need.]

5. Let $T : V \to W$ be a linear transformation from the finite-dimensional vector space V over $\mathbb{R}$ into the finite-dimensional vector space W over $\mathbb{R}$. Define the terms *rank* and *nullity* of T, and state and prove a theorem connecting them with $\dim V$.

If $V = W$, deduce that λ is an eigenvalue of T if and only if $\operatorname{rank}(T - \lambda I) < \dim V$. Show that 0 and 4 are eigenvalues of the matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and find its eigenvectors. [Mods 1994]

Supplementary Questions

6. Define $S : M_{n\times n}(\mathbb{R}) \longrightarrow M_{n\times n}(\mathbb{R})$ by $S(A) = A^T$. Prove that S has only two distinct eigenvalues and that its eigenvectors span $M_{n\times n}(\mathbb{R})$.

7. Let V be a real n-dimensional vector space, and $T : V \to V$ a linear mapping. Show that if λ is the only eigenvalue of T and T is diagonalisable then $T = \lambda I$.

Now let V be the vector space of real polynomials in x of degree at most d where $d > 0$. Which of the following linear mappings of V into itself are diagonalisable?

(i) T_1: $f(x) \mapsto x\frac{\mathrm{d}f}{\mathrm{d}x}(x)$;

(ii) T_2: $f(x) \mapsto \frac{\mathrm{d}f}{\mathrm{d}x}(x)$;

(iii) T_3: $f(x) \mapsto f(x+1)$;

(iv) T_4: $f(x) \mapsto f(-x)$.

G.A.S.