Honour Moderations: Linear Algebra
Problem Sheet 5

Michaelmas Term 2004

1. (a) Determine whether the following sets of vectors in $\mathbb{R}^4$ are linearly dependent or independent:

$$\text{(i)} \quad \begin{pmatrix} 1 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 7 \\ 1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 7 \\ -4 \end{pmatrix};$$

$$\text{(ii)} \quad \begin{pmatrix} 1 \\ 3 \\ 1 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 5 \\ -1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 3 \\ 7 \\ -2 \end{pmatrix}.$$

(b) Consider the subspaces

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \;\middle|\; b - 2c + d = 0 \right\} \quad \text{and } U = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \;\middle|\; a = d,\ b = 2c \right\}.$$

Find a basis and dimension for (i) U, (ii) W, and (iii) $U \cap W$.

2. Defining $M_{n\times n}(\mathbb{R})$ to be the set of all $n \times n$ real matrices, compute the dimensions of the following vector spaces:

(a) the subspace of $M_{n\times n}(\mathbb{R})$ consisting of all diagonal matrices (a matrix $\{a_{i,j}\}$ is *diagonal* if $a_{i,j} = 0$ for $i \neq j$);

(b) the subspace of $M_{n\times n}(\mathbb{R})$ consisting of all matrices of zero trace (that is, where the sum of the diagonal entries is zero).

3. For each of the following statements about subspaces X, Y, Z of a vector space V **either** give a proof of the statement, **or** find a counterexample. [$\mathbb{R}^2$ and $\mathbb{R}^3$ will provide all the counterexamples required.]

(a) $V \setminus X$ is never a subspace of V;

(b) $(X \cap Y) + (X \cap Z) = X \cap (Y + Z)$;

(c) $(X + Y) \cap (X + Z) = X + (Y \cap Z)$;

(d) if $Y \subseteq X$, then $Y + (X \cap Z) = X \cap (Y + Z)$.

4. Let $V = M_{2\times 2}(\mathbb{R})$, the set of all 2×2 real matrices, and let

$$U = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \;\middle|\; \text{for all } a, b \in \mathbb{R} \right\}, \quad W = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \;\middle|\; \text{for all } a, c \in \mathbb{R} \right\}.$$

Describe $U + W$ and $U \cap W$. Find bases for the subspaces U, W, $U + W$ and $U \cap W$ and hence find the dimensions of these four subspaces.

5. Let X and Y be subspaces of a finite-dimensional vector space V.

(a) Show that if X is a subspace of Y and if $\dim X = \dim Y$, then $X = Y$.

(b) Suppose that $\dim V = \dim X + \dim Y$. Show that $V = X + Y$ if and only if $X \cap Y = \{0\}$.

(c) Suppose that S and T are bases for X and Y respectively, and that $V = X + Y$, $X \cap Y = \{0\}$. Show that $S \cup T$ is a basis for V. [When $V = X + Y$ and $X \cap Y = \{0\}$, then we say that V is the *direct sum* of X and Y, and write $V = X \oplus Y$.]

(d) Show that $M_{n\times n}(\mathbb{R})$ has a basis with the property that each matrix in the basis is either symmetric or skew-symmetric.

G.A.S.