Honour Moderations: Linear Algebra
Problem Sheet 7 (Vacation Sheet)

Michaelmas Term 2004

1. Find the row rank and the column rank of

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

2. Show that if the $n \times n$ matrix A with real entries is in reduced echelon form, then **either** A has at least one zero row **or** $A = I_n$, the $n \times n$ identity matrix.

3. Let V be a vector space over $\mathbb{R}$ and U and W be subspaces of V. Define

$$U + W \;=\; \{v \in V \mid v = u + w \text{ for some } u \in U \text{ and } w \in W\}.$$

Show that $U+W$ is a subspace of V. In the case where V has dimension n ($n \geq 2$) and U and W have dimension $n-1$, show that **either** $U = W$ **or** $\dim(U \cap W) \;=\; n-2$.

4. Let V be a vector space of dimension n over $\mathbb{R}$. Prove that for each r such that $0 \leq r \leq n$, V contains a subspace of dimension r.

5. Let U be the space of all 3×3 symmetric matrices, and W the space of all 3×3 lower triangular matrices (that is, matrices of the form $C = (c_{ij})$ with $c_{ij} = 0$ unless $i \geq j$). What is $U \cap W$? Find a basis for $U + W$ which contains a basis for U and a basis for W.

6. Let V and W be finite-dimension real vector spaces, and $T : V \to W$ be a linear transformation. Define the *kernel*, $\ker(T)$, and the *image*, $\mathrm{Im}\,(T)$, of T.
Prove that T is one-to-one if and only if $\ker(T) = \{0\}$.
Suppose that $\dim V \;=\; \dim W$. Prove that T maps V onto W if and only if T is one-to-one.

7. Let A be an element of $M_{n\times n}(\mathbb{R})$, the $n \times n$ matrices with real entries. Prove that there is a polynomial $f(t) \;=\; a_r t^r + \cdots + a_1 t + a_0$, where $a_i \in \mathbb{R}$, which has A as a root and is not identically zero; that is,

$$a_r A^r + a_{r-1} A^{r-1} + \cdots + a_1 A + a_0 \;=\; 0.$$

[**Hint:** consider the matrices $I,\ A,\ A^2, \ldots$ in $M_{n\times n}(\mathbb{R})$.]

8. Let A be an $m \times n$ matrix in $M_{m\times n}(\mathbb{R})$. Prove that

(i) the space of solutions of the system $A\mathbf{x} \;=\; \mathbf{0}$ has dimension at least $n - m$;

(ii) the equation $A\mathbf{x} \;=\; \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n$ if and only if $\operatorname{rank} A \;=\; \operatorname{rank}\,[A \mid \mathbf{b}\,]$, where $[A \mid \mathbf{b}\,]$ denotes the augmented matrix with $\mathbf{b}$ as the final column.

G.A.S.