Hyperelliptic Curves

Thesis submitted in accordance
with the requirements of the
University of Liverpool
for the degree of
Doctor in Philosophy

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February 2001[1]

[1] `Header: /lfs/agstubbs/cvsroot/thesis.tex,v 1.42 2001/02/24 12:27:48 agstubbs Exp`

Abstract

This dissertation investigates hyperelliptic curves, with an emphasis on curves of genus 3. The main goal is to describe an embedding of the Kummer Variety in terms of the points in the support of the divisor representing the corresponding member of the Jacobian. Also of interest is the set of defining equations of this variety. The algebra grows very fast with the genus, and a required component of this work is the development of techniques to simplify the algebra, and a substantial amount of programming in algebra software packages, such as Maple. The algorithms here will in principle apply to hyperelliptic curves of any genus, but the size of the algebra is such that we shall only implement them in genus 3.

Acknowledgements

I would like to thank my supervisor Victor Flynn for proposing the work in this thesis and for his encouragement and supervision.

I would also like to thank Everett Howe and Michael Stoll for helpful discussions by e-mail concerning some of the ideas in Section 2.1.

Thanks also to Paul Holtom, Helen Hulme, Clare Baines, Neil Kirk, Emma Hacking, Richard Hadji, Martyn Amos, John Humphreys, Peter Giblin, Fredrik Hagen, Jenny Ölund, Bernard Kelly, Elisabeth Haugvaldstad, Ayesha, Keith and Jane Garrett, Andrew Douglas, Martin Grove, David Ashby, and all at Netcentric Systems PLC.

I acknowledge the Engineering and Physical Science Research Council (EPSRC) for financial support.

To my parents.

Contents

Chapter 1

Introduction

We shall be considering a hyperelliptic curve of genus g defined over a field K of characteristic zero.

$$\mathcal{C} : y^2 = F(x) = f_n x^n + \ldots + f_0, \tag{1.1}$$

where $n = 2g+1$ or $2g+2$, and $F(x)$ has nonzero discriminant. We shall assume throughout that either the genus is even or n is odd (or both). Then there is an identification (up to a certain non-uniqueness) between points on the Jacobian and sets of g points on the curve. More precisely, each divisor class modulo linear equivalence contains a divisor of the form

$$D = P_1 + \ldots + P_g - g\infty, \tag{1.2}$$

when n is odd, or

$$D = P_1 + \ldots + P_g - (g/2)(\infty^+ + \infty^-), \tag{1.3}$$

when the genus is even. We wish to describe an embedding for the Kummer Variety and Jacobian Variety of a hyperelliptic curve of genus 3 into projective space, in terms of the x and y coordinates of the points $P_1, \ldots, P_g$. This is theoretically possible, as proved in a theorem of Lefschetz ([22], p.105), however explicitly it is not a trivial task.

The best place to start is to examine methods used in lower genus. To date, there are only two methods which have been employed to find the analogous embedding for a hyperelliptic curve of genus 2. David Grant uses $L(3\Theta)$ to obtain an embedding into $\mathbf{P}^8$, but this is only useful for quite a restricted type of genus 2 curve: $y^2 = F(x)$ where F is some quintic polynomial in x over the ground field. Cassels and Flynn [8] use $L(4\Theta)$ to obtain an embedding into $\mathbf{P}^{15}$ for any hyperelliptic curve of genus 2.

Apart from the obvious restriction on the types of curve that Grant's embedding [21] applies to, his method also suffers from depending upon there being 'enough' second derivatives of the Weierstrass $\wp$ function. This works for genus 2, but in genus 3 we are left short. Flynn's method does not suffer from this problem, but relies heavily on spotting relations between symmetric polynomials in a style which is impractical when more than 2 coordinates are involved.

A further problem that Grant's method suffers from is that he has used the book by Baker [2] to obtain explicit expressions for the required functions, but equivalent results for genus 3 are not so readily available.

So far, my solution to the problem has been to simplify and generalize the genus 2 method of Cassels and Flynn [8] as far as possible, but (when the genus g is even) only in the slightly simpler context of $y^2 = F(x)$ where F is a polynomial of degree $2g + 1$ in x over the ground field. There are a few basic expressions, readily obtainable for any genus, from which we expect that the basis can be obtained.

The method, using $L(n\Theta)$ to obtain an embedding of a hyperelliptic genus g curve into $\mathbf{P}^{n^g-1}$, is constructive insofar as that if a symmetric polynomial in g variables has the correct weight (to be defined later), $w \leq n$, then it is either already obtainable from previously determined basis elements, or else is a newly discovered member. If a rational expression $\frac{a}{b}$ is too heavy then it may be possible to reduce the weight by considering a 'mod' b. In univariate expressions this is obviously an application of the Euclidean Algorithm, but in multivariate expressions, there is no unique way of tackling the problem. The approach will be first to define a set of rational functions in $x_1, \ldots x_g, y_1, \ldots, y_g$, consisting of the a_i , which span the regular symmetric functions in $x_1, \ldots x_g$, and the b_i are the coefficients of the polynomial passing through the (x_i, y_i). One then searches for polynomials in the a_i, b_i which give a basis for a given $L(n\Theta)$. We shall use this first to show how the genus 2 objects in [8] can be more simply derived, and then proceed to genus 3.

I started by examining symmetric polynomials, developing a couple of algorithms; one to convert a given symmetric polynomial in 3 variables into an equivalent expression in terms of 'elementary' symmetric polynomials, and another more complex algorithm to perform the same task for 3 pairs of variables. At this point it appears that this is in fact only necessary to provide 'simplified' expressions for human-readable output, or possibly to reduce the workload (and hence CPU time) for other algorithms used in the process.

I have also implemented a multivariate division algorithm, as described in Cox, Little, O'Shea. This is the canonical generalization of the Euclidean Algorithm, and will be used especially in reducing weights as described above. I have made my approach as structured as possible in terms of pre-existing computational symbolic algebra and have avoided reinventing the wheel wherever possible, which has involved some research into this not-directly-related field.

I have attempted to use quite a few programming languages to implement these algorithms; C & PARI [3], Java, Maple V R3, KASH [11] and [5], amongst a host of other less successful attempts. At the moment I have settled on Maple, but I do not particularly like the language and am continually vexed by the policy of Waterloo Software in releasing backwards-incompatible versions of Maple V. The computing situation, though greatly improved over the past few years, is still far from ideal.

Chapter 2

Geometry

We shall describe some general ideas concerning the theta divisor, and the embeddings of $L(n\Theta)$ which can be used to give the Kummer and Jacobian as projective varieties.

2.1 Geometric Background

It is instructive first to consider what happens in genus 1 and genus 2. First consider the elliptic curve

$$\mathcal{E} : y^2 = x^3 + g_2x^2 + g_1x + g_0, \tag{2.1}$$

defined over a field K of characteristic zero, where the cubic has nonzero discriminant. We shall take ∞ to be the identity. There is the standard correspondence between points on $\mathcal{E}$ and members of the Jacobian, where $P \in \mathcal{E}$ corresponds to the divisor class containing $P - \infty$. Define the standard weights function $o_n(f)$ by considering the order of the pole of f at ∞ on the curve; on $\mathcal{E}$, define

$$o_n(x) = 2,\ \ o_n(y) = 3. \tag{2.2}$$

Now, consider the bases for functions of weight at most 2, at most 3 and at most 4:

$$\begin{gathered} \kappa_0 = 1,\ \ \kappa_1 = x, \\ \rho_0 = 1,\ \ \rho_1 = x,\ \ \rho_2 = y, \\ z_0 = 1,\ \ z_1 = x,\ \ z_2 = y,\ \ z_3 = x^2. \end{gathered} \tag{2.3}$$

We can use either the second or third bases to give the coordinates of a projective embedding of the Jacobian variety. In the case of ρ_0, ρ_1, ρ_2, we have as the defining equation merely the homogenisation of (2.1).

$$\mathcal{E} : \rho_0 \rho_2^2 = \rho_1^3 + g_2 \rho_0 \rho_1^2 + g_1 \rho_0^2 \rho_1 + g_0 \rho_0^3. \tag{2.4}$$

In the case of z_0, z_1, z_2, z_3, we have a pair of defining equations

$$\begin{gathered} z_2^2 = z_1 z_3 + g_2 z_0 z_3 + g_1 z_0 z_1 + g_0 z_0^2, \\ z_1^2 = z_0 z_3, \end{gathered} \tag{2.5}$$

which give the Jacobian as the intersection of quadratic forms, as a special case of [26]. Note that here, with genus $g = 1$, there are $2^g, 3^g, 4^g$ members of the bases in (2.3), and that the defining equations for the ρ_i are cubic, and the defining equations for the z_i are quadratic. Also, note that, of the z_i, $1 = (4^g - 2^g)/2$ is *odd* (negated by $P \mapsto -P$) and $3 = (4^g + 2^g)/2$ are *even* (invariant under $P \mapsto -P$).

In genus 2, let

$$\mathcal{C} : y^2 = f_6 x^6 + f_5 x^5 + \ldots + f_0, \tag{2.6}$$

defined over a field K of characteristic zero, where the sextic has nonzero discriminant. When $f_6 \neq 0$, we use ∞^+, ∞^- to denote the points on the non-singular curve, the normalisation of $\mathcal{C}$, that lie over the singular point at infinity on $\mathcal{C}$ (for any curve (2.6) with $f_6 \neq 0$ both ∞^+ and ∞^- are in $\mathcal{C}(K)$ when $f_6 \in (K^*)^2$). When $f_6 = 0$ (and so $f_5 \neq 0$), we let ∞ denote the point at infinity, which is always in $\mathcal{C}(K)$. Following Chapter 1 of [8], any member of $\mathcal{J}(K)$, the K-rational points on the Jacobian, may be represented by a divisor of the form $P_1 + P_2 - \infty^+ - \infty^-$, where P_1, P_2 are points on $\mathcal{C}$ and either P_1, P_2 are both K-rational or P_1, P_2 are quadratic over K and conjugate. We shall abbreviate such a divisor by: $\{P_1, P_2\}$. This representation gives a 1-1 correspondence with members of $\mathcal{J}(K)$, except that everything of the form $\{(X,Y),(X,-Y)\}$ must be identified into a single equivalence class $\mathcal{O}$, which serves as the group identity in $\mathcal{J}(K)$. Note that $-\{(x_1,y_1),(x_2,y_2)\} = \{(x_1,-y_1),(x_2,-y_2)\}$; furthermore $\{P_1,P_2\} + \{Q_1,Q_2\} + \{R_1,R_2\} = \mathcal{O}$ if and only if there exists $\Upsilon(X)$ of degree ≤ 3 such that $Y = \Upsilon(X)$ meets $\mathcal{C}$ at $P_1, P_2, Q_1, Q_2, R_1, R_2$. These two rules are sufficient for computing the group law on $\mathcal{J}(K)$. Clearly, an element of order 2 in $\mathcal{J}(K)$ is given by $\{(X_1,0),(X_2,0)\}$, where X_1, X_2 are the roots of quadratic $Q(X)$ defined over K, satisfying $Q(X)|F(X)$.

We define

$$o_n(x) = 2,\ o_n(y) = \ 5 \text{ (if } f_6 = 0) \text{ or } 6 \text{ (if } f_6 \neq 0). \tag{2.7}$$

Following a similar route to that exposed in Cassels and Flynn [8], we first look at functions in x_1, y_1, x_2, y_2 regular away from $\mathcal{O}$ with order of pole at infinity of at most 2; that is to say, a set of coordinates for the *Kummer surface*. By this we mean that, within each monomial, we consider the weight in (x_1, y_1) and (x_2, y_2) separately, and take the maximum. For example, the weight (or order at infinity) of $x_1^3 x_2$ is 6 (since the weight of x_1^3 is 6 and that of x_2 is 2). For rational functions, the weight is just the weight of the numerator minus that of the denominator. We can also think of this as a basis for $L(\Theta^+ + \Theta^-)$, where Θ^+ and Θ^- are the images of the curve in the Jacobian given by the images of $P \mapsto P - \infty^+$ and $P \mapsto P - \infty^-$, respectively, These are given in Chapter 2 of [8] by

$$\kappa_0 = 1,\ \kappa_1 = (x_1 + x_2),\ \kappa_2 = x_1 x_2,\ \kappa_3 = \frac{F_0(x_1, x_2) - 2y_1 y_2}{(x_1 - x_2)^2}, \tag{2.8}$$

where

$$\begin{aligned} F_0(x_1, x_2) = \;& 2f_0 + f_1(x_1 + x_2) + 2f_2(x_1 x_2) + f_3(x_1 + x_2)(x_1 x_2) \\ & +2f_4(x_1 x_2)^2 + f_5(x_1 + x_2)(x_1 x_2)^2 + 2f_6(x_1 x_2)^3. \end{aligned} \tag{2.9}$$

Note that there are $4 = 2^g$ members of this basis $\kappa_0, \ldots, \kappa_3$ of $L(\Theta^+ + \Theta^-)$ when $f_6 \neq 0$, and the same definitions of $\kappa_0, \ldots, \kappa_3$ give a basis for $L(2\Theta)$ when $f_6 = 0$.

When $f_6 = 0$, Grant [21] has described a basis $\rho_0, \ldots, \rho_8$ for $L(3\Theta)$, which we do not reproduce here. This gives an embedding of the Jacobian variety in $\mathbf{P}^8$. Again note that there are $9 = 3^g$ members of this basis for $L(3\Theta)$.

When $f_6 \neq 0$, $L(3\Theta)$ is not defined over the ground field, and we must use $L(2(\Theta^+ + \Theta^-))$, which is geometrically the same as $L(4\Theta)$ when $f_6 = 0$. We reproduce here the basis $z_0, \ldots, z_{15}$ in Chapter 2 of [8].

$$\begin{gathered} z_{15} = (x_1 - x_2)^2,\ z_{14} = 1,\ z_{13} = x_1 + x_2,\ z_{12} = x_1 x_2, \\ z_{11} = x_1 x_2 (x_1 + x_2),\ z_{10} = (x_1 x_2)^2,\ z_9 = (y_1 - y_2)/(x_1 - x_2), \\ z_8 = (x_2 y_1 - x_1 y_2)/(x_1 - x_2),\ z_7 = (x_2^2 y_1 - x_1^2 y_2)/(x_1 - x_2), \\ z_6 = (x_2^3 y_1 - x_1^3 y_2)/(x_1 - x_2),\ z_5 = (F_0(x_1, x_2) - 2y_1 y_2)/(x_1 - x_2)^2, \\ z_4 = (F_1(x_1, x_2) - (x_1 + x_2) y_1 y_2)/(x_1 - x_2)^2,\ z_3 = (x_1 x_2) z_5, \\ z_2 = (G_0(x_1, x_2) y_1 - G_0(x_2, x_1) y_2)/(x_1 - x_2)^3, \\ z_1 = (G_1(x_1, x_2) y_1 - G_1(x_2, x_1) y_2)/(x_1 - x_2)^3,\ z_0 = z_5^2. \end{gathered} \tag{2.10}$$

where

$$\begin{aligned}
F_0(x_1,x_2) = \quad & 2f_0 + f_1(x_1+x_2) + 2f_2(x_1x_2) + f_3(x_1x_2)(x_1+x_2) \\
& +2f_4(x_1x_2)^2 + f_5(x_1x_2)^2(x_1+x_2) + 2f_6(x_1x_2)^3, \\
F_1(x_1,x_2) = \quad & f_0(x_1+x_2) + 2f_1(x_1x_2) + f_2(x_1x_2)(x_1+x_2) + 2f_3(x_1x_2)^2 \\
& +f_4(x_1x_2)^2(x_1+x_2) + 2f_5(x_1x_2)^3 + f_6(x_1x_2)^3(x_1+x_2),
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
G_0(x_1,x_2) = \quad & 4f_0 + f_1(x_1+3x_2) + f_2(2x_1x_2+2x_2^2) + f_3(3x_1x_2^2+x_2^3) \\
& +4f_4(x_1x_2^3) + f_5(x_1^2x_2^3+3x_1x_2^4) + f_6(2x_1^2x_2^4+2x_1x_2^5), \\
G_1(x_1,x_2) = \quad & f_0(2x_1+2x_2) + f_1(3x_1x_2+x_2^2) + 4f_2(x_1x_2^2) + f_3(x_1^2x_2^2+3x_1x_2^3) \\
& +f_4(2x_1^2x_2^3+2x_1x_2^4) + f_5(3x_1^2x_2^4+x_1x_2^5) + 4f_6(x_1^2x_2^5).
\end{aligned} \tag{2.12}$$

There are $16 = 4^g$ members of this basis, which also gives an embedding of the Jacobian variety. Note that $6 = (4^g - 2^g)/2$ are *odd* (negated by $P \mapsto -P$) and $10 = (4^g + 2^g)/2$ are *even* (invariant under $P \mapsto -P$). The canonical divisor class $\mathcal{O}$ is mapped to $(1, 0, \ldots, 0)$, and the Mordell-Weil group $\mathrm{Pic}^0_K(\mathcal{C})$ is mapped into $\mathbf{P}^{15}(K)$.

The defining equation for the Kummer surface is given by the quartic equation in $\mathbf{P}^3$,

$$R(\kappa_0,\kappa_1,\kappa_2)\kappa_3^2 + S(\kappa_0,\kappa_1,\kappa_2)\kappa_3 + T(\kappa_0,\kappa_1,\kappa_2) = 0, \tag{2.13}$$

where R, S, T are given by:

$$\begin{aligned}
R(\kappa_0,\kappa_1,\kappa_2) &= \kappa_1^2 - 4\kappa_0\kappa_2, \\
S(\kappa_0,\kappa_1,\kappa_2) &= -2\left(2f_0\kappa_0^3 + f_1\kappa_0^2\kappa_1 + 2f_2\kappa_0^2\kappa_2 + f_3\kappa_0\kappa_1\kappa_2 + 2f_4\kappa_0\kappa_2^2\right. \\
&\quad \left. +f_5\kappa_1\kappa_2^2 + 2f_6\kappa_2^3\right), \\
T(\kappa_0,\kappa_1,\kappa_2) &= -4f_0f_2\kappa_0^4 + f_1^2\kappa_0^4 - 4f_0f_3\kappa_0^3\kappa_1 - 2f_1f_3\kappa_0^3\kappa_2 - 4f_0f_4\kappa_0^2\kappa_1^2 \\
&\quad +4f_0f_5\kappa_0^2\kappa_1\kappa_2 - 4f_1f_4\kappa_0^2\kappa_1\kappa_2 - 4f_0f_6\kappa_0^2\kappa_2^2 + 2f_1f_5\kappa_0^2\kappa_2^2 \\
&\quad -4f_2f_4\kappa_0^2\kappa_2^2 + f_3^2\kappa_0^2\kappa_2^2 - 4f_0f_5\kappa_0\kappa_1^3 + 8f_0f_6\kappa_0\kappa_1^2\kappa_2 - 4f_0f_6\kappa_1^4 \\
&\quad -4f_1f_5\kappa_0\kappa_1^2\kappa_2 + 4f_1f_6\kappa_0\kappa_1\kappa_2^2 - 4f_2f_5\kappa_0\kappa_1\kappa_2^2 - 2f_3f_5\kappa_0\kappa_2^3 \\
&\quad -4f_1f_6\kappa_1^3\kappa_2 - 4f_2f_6\kappa_1^2\kappa_2^2 - 4f_3f_6\kappa_1\kappa_2^3 - 4f_4f_6\kappa_2^4 + f_5^2\kappa_2^4.
\end{aligned}$$

The defining equations for Grant's $L(3\Theta)$ embedding are given by equations that are at most cubic, and the defining equations for the $L(4\Theta)$ embedding are give by quadratic equations.

We now generalise the above patterns and definitions, and give some standard results about dimensions and embeddings.

Definition 2.1 *For $\mathcal{C}$ as in (1.1) with n odd, define Θ to be the set of all divisors of the forms: $(x_1, y_1) - \infty$, $(x_1, y_1) + (x_2, y_2) - 2\infty$, $\ldots$, $(x_1, y_1) + .. + (x_{g-1}, y_{g-1}) - (g-1)\infty$, as described in [25], 11.2.*

When the genus is 2, $L(2\Theta)$ gives an embedding of the *Kummer surface.* It is natural to generalise this as follows.

Definition 2.2 *For any hyperelliptic curve $\mathcal{C}$, we shall refer to the locus of a basis of $L(2\Theta)$ as a* Kummer variety*.*

By [25], 7.3, 7.4, 11.1 and [26], p.163, we have the following.

Lemma 2.3 *Let Θ be as in Definition (2.1). Then Θ is ample, and $L(n\Theta)$ for $n \geq 3$ will be very ample, and so induces a projective embedding of the Jacobian. The defining equations for the $L(3\Theta)$ embedding of the Jacobian variety are all at most cubic, and the defining equations for the $L(4\Theta)$ embedding of the Jacobian variety are all quadratic.*

By analogy with genus 2, where the defining equation is a quartic (perhaps provable by generalising the argument in 10.3, although we have not been able to do so), we make the following conjecture.

Conjecture 2.4 *Let Θ be as in Definition (2.1). The defining equations for the $L(2\Theta)$ embedding of the Kummer variety are given by equations of degree at most four.*

Note that this is compatible with the fact that the defining equations for the $L(4\Theta)$ embedding are all quadratic, since any such quadratic equation in the even functions of $L(4\Theta)$ induces a quartic equation in $L(2\Theta)$.

Lemma 2.5 *Flynn [18], and Grant [21] both state the well-known result that the dimension of $L(n\Theta)$ is n^g in genus g.*

Lemma 2.6 *From [25], pp. 57,58,66,93, we know that there is a basis of $L(4\Theta)$ consisting of $\frac{1}{2}(4^g + 2^g)$ even functions and $\frac{1}{2}(4^g - 2^g)$ odd functions.*

A natural problem now is to try to find objects analogous to the κ_i, ρ_i and z_i above, in the case of genus 3. One can see from the above how quickly the quantity of algebra explodes in going from genus 1 to 2, and there is a similar further explosion in the size of the expression in going from genus 2 to genus 3. Therefore there will be severe computational constraints on what is possible.

2.2 Theta Functions

Our techniques will not rely on the theory of theta functions. Nevertheless, for the sake of context, we shall give a brief overview, broadly following [2] but with the notation slightly modernised. We shall assume, as is common in analytic treatments, that the curve (1.1) is in 'canonical' form, with $f_{2g+2} = 0$ and $f_{2g+1} = 4$.

Remember that, for a given hyperelliptic curve, $\mathcal{C}$, of genus g, there is a lattice $\Omega_{\mathcal{C}}$ such that $\mathbb{C}^g/\Omega_{\mathcal{C}}$ is homeomorphic, by Abel's Theorem, to the Jacobian of $\mathcal{C}$.

Definition 2.7 *A theta function on this lattice is a holomorphic function which is pseudo-periodic with respect to the lattice Ω, often defined as*

$$\vartheta(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i n^t \Omega n + 2\pi i n^t z) \tag{2.14}$$

A slight generalization of ϑ are the theta functions with rational characteristics $\vartheta[a,b](z,\Omega)$ with $a,b \in \mathbb{Q}^g$.

Riemann has shown that there are $a,b,\in \mathbb{Z}$ s.t. $\vartheta[a,b](z)$ has a zero of order one precisely along the pullback of Θ to $\mathcal{C}^g$. This is the specific theta function, ϑ which we are particularly interested in, as the second derivatives of $\ln\vartheta$ give rise to the function known as the Weierstrass $\wp$-function. We denote $\zeta_i = \frac{\partial \ln\vartheta}{\partial u_i}$, where u_i is a coordinate on $\mathbb{C}^g/\Omega_{\mathcal{C}}$.

We give the following inductive definition.

Definition 2.8

$$\begin{aligned} \wp_{ij}(u) &= -\tfrac{\partial \zeta_i}{\partial u_j}(u) \\ \wp_{i\ldots jk} &= -\tfrac{\partial}{\partial u_k}\wp_{i\ldots j}(u) \end{aligned} \tag{2.15}$$

For notational convenience we also make the following definition.

Definition 2.9

$$\ell_i(s,t) = \frac{1}{4t}\sum_{k=i}^{2g+1-i}(k+1-i)f_{k+1+i}s^k, 1 \le i \le g. \tag{2.16}$$

Lemma 2.10 *The canonical basis for the differentials of the second kind is given by $\ell_i(s,t)ds$, where $t^2 = F(s)$.*

Proof See [1] §138, especially Ex. (i). □

Definition 2.11 *For $1 \le i \le g$ and $t^2 = F(s)$ we define the integrals of the second kind L_i:*

$$L_i^{P,Q} = \int_Q^P \ell_i(s,t)ds. \tag{2.17}$$

Lemma 2.12 *Using the convention that* $\frac{\partial x_k}{\partial u_i} = 0$ *for* $i > g$ *we find the following convenient expression*

$$\zeta_i(u) = -\sum_{k=1}^{g} L_i^{P_k,Q_k} + \frac{1}{2}\sum_{k=1}^{g} \frac{\partial x_k}{\partial u_{i+1}} \tag{2.18}$$

Proof See [1] §215, especially Ex. (vi). □

Lemma 2.13 *We can express the hyperelliptic* $\wp$ *functions as rational functions of the variables* $x_1, \ldots, x_g, y_1, \ldots, y_g$ *on* $S^g\mathcal{C}$*.*

$$\wp_{ij}(u) = \sum_{k=1}^{g} \ell_i(x_k, y_k)\frac{\partial x_k}{\partial u_j} - \frac{1}{2}\sum_{k=1}^{g} \frac{\partial^2 x_k}{\partial u_j \partial u_{i+1}}. \tag{2.19}$$

Proof Differentiate (2.18) with respect to u_j. □

In order to calculate $\frac{\partial x_k}{\partial u_i}$ we need to work with the system of differential equations

$$\mathrm{d}u_i = \sum_{k=1}^{g} x_k^{i-1}\frac{\mathrm{d}x_k}{y_k}, 1 \le i \le g, \tag{2.20}$$

Define V to be the Vandermonde matrix for $x_1, \ldots, x_g$ to obtain

$$(\mathrm{d}u_1, \ldots, \mathrm{d}u_g) = \Big(\frac{\mathrm{d}x_1}{y_1}, \ldots, \frac{\mathrm{d}x_g}{y_g}\Big)V. \tag{2.21}$$

V is nonsingular, and so we can express $\frac{\mathrm{d}x_i}{y_i}$ in terms of the $\mathrm{d}u_j$. Hence we have the following lemma.

Lemma 2.14 *Define* V *to be the Vandermonde matrix for* $x_1, \ldots, x_g$*. Then, denoting the ith row and jth column in* V^{-1} *by* $V_{i,j}^{-1}$ *we have*

$$\frac{\partial x_i}{\partial u_j} = y_i V_{j,i}^{-1}. \tag{2.22}$$

In [21] second derivatives of logs of theta functions are used in genus 2 to get bases for such objects as $L(2\Theta)$. In genus 3, it must be possible in principle to use theta functions, at least in the $y^2 =$ odd degree case, but it seems to be much less straightforward than in genus 2 to find bases expressed in terms of the x_i and y_i. For example, in genus 3 there is no way to determine $L(2\Theta)$ which has dimension 8 purely from second derivatives of the log of a theta function, of which there are 6. One needs some fairly exotic polynomial in these 6 functions to fill out $L(2\Theta)$. Our approach will be entirely independent of theta functions, and will rely essentially only on linear algebra.

Chapter 3

Kummer and Jacobian Varieties for Curves of Genus 3

We shall consider the problem of trying to find a basis for $L(n\Theta)$ in terms of the x_i, y_i that occur in the support of $(x_1, y_1) + \ldots + (x_g, y_g) - g \cdot \infty$. More detailed descriptions of some of the algorithms will be in the next chapter. We shall first require some preliminaries on symmetric polynomials. We then proceed to define some weights which are required to simplify computations. We show how to obtain an elementary rederivation of the basis elements in genus 2, and then give our main result: a basis for $L(2\Theta)$ in genus 3.

3.1 Outline of General Strategy

For a set of g points $(x_1, y_1), \ldots, (x_g, y_g)$ on a hyperelliptic curve (1.1) of genus g, define a_i and b_i as follows.

Definition 3.1 *Let $a_0, \ldots, a_g$ be the unique polynomials in $x_1, \ldots, x_g$ satisfying*

$$(x + x_1)(x + x_2) \ldots (x + x_g) = a_0 x^g + a_1 x^{g-1} + \ldots + a_g. \tag{3.1}$$

Let $b_0, \ldots, b_{g-1}$ be the unique rational functions in $x_1, \ldots, x_g, y_1, \ldots, y_g$ satisfying

$$y_i = b_0 x_i^{g-1} + b_1 x_i^{g-2} + \ldots + b_{g-1}, \text{ for all } i \text{ satisfying } 1 \le i \le g. \tag{3.2}$$

The polynomials $a_0, \dots, a_g$ are often referred to as the *elementary symmetric polynomials* in $x_1, \dots x_g$. Any symmetric polynomial in $x_1, \dots x_g$ is a polynomial in the elementary symmetric polynomials. The rational functions $b_0, \dots b_g$ have denominator

$$\delta = \det\bigl(x_j^{i-1}\bigr). \tag{3.3}$$

An affine piece of the Jacobian can be represented by the $(2g-1)$-tuples $[a_0, \dots, a_g, b_0, \dots, b_{g-1}]$. However, this way of representing members of the Jacobian does not immediately give us all of the members of a given $L(n\Theta)$. To take a simple example, when $g = 2$, we know from Lemma 2.5 that the dimension of $L(2\Theta)$ is 4, whereas only a_0, a_1, a_2 (and no other a_i and no b_i) are members of $L(2\Theta)$, so that one basis member is missing. For higher genus and higher values of n, the situations becomes worse, with progressively smaller and smaller proportions of the n^g basis members for $L(n\Theta)$ appearing directly on the list $a_0, \dots, a_g, b_0, \dots, b_{g-1}$.

Our strategy is to obtain missing members of a given $L(n\Theta)$ as polynomials in $a_0, \dots, a_g, b_0, \dots, b_{g-1}$. We first generalise the weight of (2.2),(2.7).

Definition 3.2 *Let $\mathcal{C} : y^2 = f_n x^n + \dots + f_0$ be a hyperelliptic curve of genus g as in (1.1). Define $o_n(x_i) = 2$, $o_n(y_i) = n$ for any i, and $o_n(f_j) = 0$. Define o_n on monomials by*

$$o_n(x_1^{i_1} \dots x_g^{i_g} y_1^{j_1} \dots y_g^{j_g}) = max\bigl(2i_1 + nj_1, \dots, 2i_g + nj_g\bigr). \tag{3.4}$$

Define o_n of a polynomial in $x_1, \dots, x_g, y_1, \dots, y_g$ to be the maximum o_n of a monomial occurring in the polynomial. Extend o_n to rational functions in $x_1, \dots, x_g, y_1, \dots, y_g$ by defining $o_n(p/q) = o_n(p) - o_n(q)$.

To find the missing members of a given $L(n\Theta)$, we seek polynomials in $a_0, \dots, a_g, b_0, \dots, b_{g-1}$ which have weights at most n. Since the expressions in the x_i, y_i are extremely large, we shall require ways of simplifying the computations, which will be described and illustrated in the following sections.

3.2 Symmetric polynomials in pairs of variables

The notion of symmetric and antisymmetric polynomials in pairs of variables is crucial in much of the work of pole-reduction. Here we examine some of the general theory which is required.

Definition 3.3 *A* symmetric polynomial in pairs of variables *is a polynomial in a given number, n, of pairs of variables (x_i, y_i) which is independent under the action of the permutation group S_n given by $\sigma : (x_i, y_i) \mapsto (x_{i\sigma}, y_{i\sigma})$. We denote the space of such functions $S_n\mathbb{F}[X, Y]$; a subspace of $\mathbb{F}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$.*

Definition 3.4 *Define an index function*

$$i_n(x, y) : \{(x, y) : 1 \leq x + y \leq n\} \twoheadrightarrow \{1, \ldots, \frac{1}{2}n(n + 3)\}$$

as follows:

$$i_n(x, y) = \frac{y(2n + 3) - y^2}{2} + x$$

This is invertible in the range of definition, with inverse given by:

$$i_n^{-1}(m) = (m - i_n(0, y), y)$$

where $0 \leq y \leq n$ is chosen so that either:

- *$y = n$ and $i_n(0, y) = m$; or*
- *$i_n(0, y) \leq m < i_n(0, y + 1)$, for $y < n$.*

Definition 3.5 *Define* elementary symmetric polynomials *in the n pairs of variables (x_i, y_i), as follows $(1 \leq x + y \leq n)$:*

$$r_{i_n(x,y)} = \sum_{p \in I_y} \left((\prod_{j \in p} Y_j)(\sum_{q \in J_{p,x}} \prod_{k \in q} X_k) \right)$$

where $I_y = \{\{1\sigma, \ldots, y\sigma\} : \sigma \in S_n\}$, $I_0 = \{\emptyset\}$, and $\forall p \in I_y$, $J_{p,x} = \{(\{1, \ldots, n\} - p)$ choose $x\}$.

Conjecture 3.6 *Any polynomial symmetric in the n pairs of variables (x_i, y_i), $1 \leq i \leq n$, may be expressed in terms of the elementary symmetric polynomials r_j, $1 \leq j \leq \frac{1}{2}n(n+3)$.*

Lemma 3.7 *The weight functions $x(r_j)$ and $y(r_j)$ are given as follows:*

$$x(r_j) = \pi_x i_n^{-1}(j)$$
$$y(r_j) = \pi_y i_n^{-1}(j).$$

Then each monomial in r_j will have a well-defined weight-pair (x, y) which allows us to partition the space of symmetric polynomials in the (x_j, y_j) into homogeneous subspaces; any polynomial expression in the r may be projected onto those spaces to obtain (x, y)-homogeneous subexpressions which when added together give the original expression.

Definition 3.8 *Denote by $H_{x,y}$ the homogeneous subspace of $S_n\mathbb{F}[X, Y]$ of weight (x, y). Also use this symbol to denote the projection map $H_{x,y} : S_n\mathbb{F}[X, Y] \twoheadrightarrow H_{x,y}[X, Y]$. Clearly,*

$$S_n\mathbb{F}[X, Y] = \bigoplus_{x,y \geq 0} H_{x,y}[X, Y].$$

3.3 Representation of Symmetric Polynomials

Given a symmetric polynomial S in variables $x_1, \ldots, x_g$, and limited processing power, we soon find ourselves looking for the minimum amount of information required such that the original S may be reconstructed as and when necessary. The initial approach may be to try to express the polynomial S in terms of the appropriate elementary symmetric polynomials, but this quickly runs into trouble for precisely the reasons that we are trying to prevent.

Example 3.9 *Given $S = x_1^3(x_2 + x_3) + x_2^3(x_1 + x_3) + x_3^3(x_1 + x_2)$, then S may be reconstructed from $x_1^3 x_2$.*

Suppose the degree of S with respect to x_1 alone is d. Then the coefficient of x_1^d is symmetric in $x_2, \ldots, x_g$. This gives rise to a recursive procedure for completely describing at least the terms in S containing x_i^d for each i.

This recursivity pretty much completely describes the algorithm required for 'normal' symmetric polynomials, and suggests a canonical representation for S:

$$c_1 x_1^{d_{1,1}} x_2^{d_{1,2}} \ldots x_g^{d_{1,g}} + \cdots + c_k x_1^{d_{k,1}} x_2^{d_{k,2}} \ldots x_g^{d_{k,g}}$$

where $d_{i_1,j_1} \geq d_{i_2,j_2}$ when either $i_1 < i_2$ and $j_1 = 1 = j_2$; or $i_1 = i_2$ and $j_1 < j_2$. If C_1 and C_2 are the canonical representations of S_1 and S_2 respectively, then $C_1 + C_2$ is the canonical representation for $S_1 + S_2$.

3.4 Representation of Hyperelliptic Expressions

Recall that, in this context, a Hyperelliptic Expression is a rational polynomial symmetric in pairs of variables (x_i, y_i), $1 \leq i \leq g$ such that the highest power of any individual y_i is exactly one.

This gives us a natural representation:

$$A = A_0 + A_{1,1}y_1 + \cdots + A_{1,g}y_g + A_{2,1}y_1y_2 + \cdots + A_{2,\frac{g!}{2!(g-2)!}}y_{g-1}y_g + \cdots$$

where $A_{i,1}$ is symmetric in $\{x_1, \ldots, x_i\}$ and $\{x_{i+1}, \ldots, x_g\}$ separately. The conditions require that the orbit of $A_{i,1}y_1 \ldots y_i$ is symmetric in the pairs (x_i, y_i) for all $1 \leq i \leq g$.

Determine the representative, $R_{i,1}$, for $A_{i,1}$ regarded as a symmetric polynomial in $\{x_1, \ldots, x_i\}$. Then, find the representative R_2 for $A_{i,1} - \text{orb } R_{i,1}$ regarded as a symmetric polynomial in $\{x_{i+1}, \ldots, x_g\}$. Then, the representative for the terms with y-weight i is simply $R_i = (R_{i,1} + R_{i,2})y_1 \ldots y_i$.

Denoting the canonical representative of A_0 by R_0 gives a canonical representation:

$$A = \text{orb} \sum_j R_j$$

	x	y	w	o_{2g+1}	o_{2g+2}
x_i	1	0	1	2	2
y_i	0	1	$g+1$	$2g+1$	$2g+2$
f_i	$-i$	2	$2g+2-i$	0	0
a_i	i	0	i	2	2
a_{g+1}	0	1	$g+1$	$2g+1$	$2g+2$
a_{g+2}	1	1	$g+2$	$2g+3$	$2g+4$
a_{2g}	$g-1$	1	$2g$	$2g+1$	$2g+2$
b_i	$1+i-g$	1	$2+i$	3	4

Figure 3.1: Weights and Measures

3.5 Weights and Measures

The techniques I have developed require us to treat expressions in the x's, y's and f's as a graded pseudo-homogeneous ring. The weights of principal importance are as indicated in Figure 3.1. Note that, in the figure the x, y, w columns are weights and the o columns are the order of the pole at ∞ for that expression. Also, $w = (g+1)y + x$.

Hereafter, we shall denote by a_n $(1 \leq n \leq g)$, as in (3.1) the positive representation of the elementary symmetric polynomials in n-tuples of the x_i's; b_i is as in (3.2), which is the corresponding coefficient in the curve given by

$$\mathcal{D} : y = b_0 x^{g-1} + \cdots + b_{g-2}x + b_{g-1} \tag{3.5}$$

which passes through each of the points P_i and intersects the curve $\mathcal{C}$ with the correct multiplicity at each of those points; we also define 'extra' symmetric polynomials as follows:

$$a_i = \begin{cases} y_1 + y_2 + \cdots + y_g & \text{when } i = g+1, \\ x_1y_1 + x_2y_2 + \cdots + x_gy_g & \text{when } i = g+2, \\ x_1x_2 \ldots x_{g-1}y_g + \cdots + x_2x_3 \ldots x_gy_1 & \text{for } i = 2g. \end{cases}$$

Conjecture 3.10 *Given a hyperelliptic curve, $\mathcal{C} : y^2 = F(x)$, of genus g, the a_i*

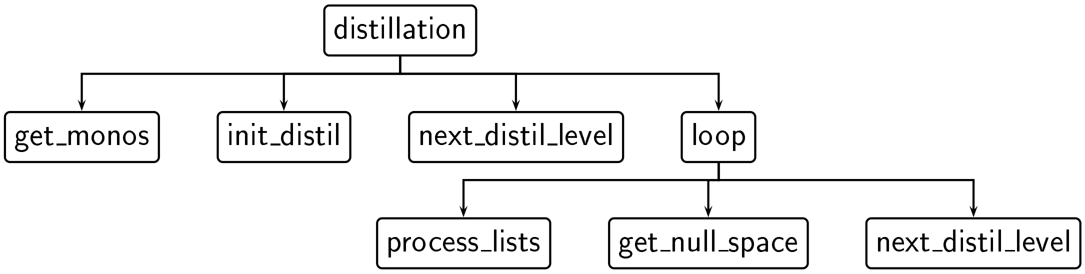

Figure 3.2: Top Down Analysis of distillation

form a basis for the space spanned by the regular symmetric polynomials in the pairs (x_i, y_i), $(1 \leq i \leq g)$, *modulo the relation defined by the underlying curve.*

The principal limitation at this point is that currently the algorithms I have implemented in `maple` take a long time to run, and sometimes the objects become too large for `maple` to cope with. it is necessary to optimize to more appropriately balance memory and CPU time considerations.

3.6 Distillation

One first decides on n for the $\mathcal{L}(n\Theta_g)$ embedding. Then choose a y-weight and a w-weight to search. Collect all appropriate monomial expressions in the a's, b's and f's and separate into levels, such that two expressions are in the same level iff they have the same o-value; i.e., the order of the pole at ∞ is the same for both. Then, using simple linear algebra, it is possible to find linear combinations that sit in a lower level (i.e., the order of the pole at ∞ is strictly less). Iterating from the highest level to the lowest level is the most efficient way of determining which combinations have a pole of order at most n. We give an overview in the following diagram.

The only problem is when n is odd; for, when $d = 2g + 2$, $\mathcal{L}(n\Theta_g)$ is empty, so this complicates finding a basis for $\mathcal{L}(n\Theta_g)$ which is independent of the degree d of the curve $\mathcal{C}$. In this case, we must initially pretend that d is odd (i.e. $d' = d - 1 = 2g + 1$) to do an initial distillation to obtain expression list 1. Then, a further distillation for n with $d = 2g + 2$, followed by throwing away any

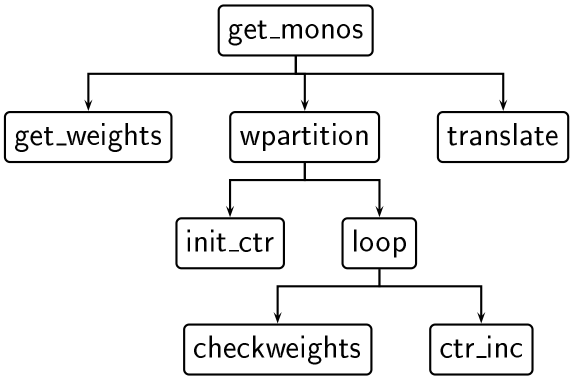

Figure 3.3: Top Down Analysis of get_monos

expression in the new list for which the coefficient of f_d is zero to obtain list 2. Finally, mixing together list 1 and list 2 and performing a final distillation on the resulting list gives the required output.

3.7 Simplified Derivation in Genus 2

Recall that in genus 2 our curve $\mathcal{C}$ is as in (2.6) and a member of the Jacobian can be described as $(x_1, y_1) + (x_2, y_2) - \infty^+ - \infty^-$. Then the a_i, b_i, δ of (3.1), (3.2), (3.3), respectively, are

$$\begin{gathered} a_0 = 1,\ a_1 = x_1 + x_2,\ a_2 = x_1 x_2, \\ b_0 = (y_2 - y_1)/(x_2 - x_1),\ b_1 = (x_2 y_1 - x_1 y_2)/(x_2 - x_1), \\ \delta = x_2 - x_1. \end{gathered} \tag{3.6}$$

As we have already observed, a_0, a_1, a_2 all have $o_n(a_i) \leq 2$ and so are all in $L(2\Theta)$. We are therefore missing one basis element. However, $o_n(b_0) = o_n(b_1) = 4$, and do not give the missing basis member. The natural next stage to investigate quadratic monomials, such as

$$b_0^2 = \big(F(x_1) + F(x_2) - 2y_1 y_2\big)/(x_1 - x_2)^2. \tag{3.7}$$

Now, $o_n(b_0^2) = 8$, which seems even worse than before. However, note that $o_n((x_1 - x_2)^2) = 4$, and so we only require that all terms in the numerator have o_n at most 6, a condition already satisfied by the $y_1 y_2$ term. Available to us is the fact that we can adjust b_0^2 by polynomials in a_0, a_1, a_2, equivalent to adjusting $F(x_1) + F(x_2)$ modulo δ^2. Now note, for example, that the terms of highest o_n are $f_6 x_1^6 + f_6 x_2^6$, which satisfy $o_n(f_6 x_1^6 + f_6 x_2^6) = 12$. But subtracting $f_6(x_1 + x_2)^4(\delta^2)$ immediately reduces this 12 to 10. This motivates the following inductive procedure.

Definition 3.11 *Let $p = p(x_1, x_2)$ be a symmetric polynomial in x_1, x_2, with highest weight terms $c(x_1^k x_2^\ell + x_1^\ell x_2^k)$ where c is a constant and, without loss of generality, we take $k \geq \ell$. If $k \geq \ell + 2$ then replace p by*

$$p - c x_1^\ell x_2^\ell (x_1 + x_2)^{k-\ell-2} \delta^2. \tag{3.8}$$

Continue this process until it is no longer the case that $k \geq \ell + 2$ for the highest weight terms. We denote the final result as $Red_{o_n}(p)$ modulo δ^2.

We can now more succinctly describe κ_3 of (2.8) as: $\kappa_3 = \mathrm{Red}_{o_n}(b_6^2)$ modulo δ^2. As a polynomial in a_0, a_1, a_2, b_0, b_1 this is

$$\kappa_3 = b_6^2 - f_2 - f_3 a_1 - f_4 a_1^2 + f_5 a_1 a_2 - f_5 a_1^3 - f_6 a_2^2 + 2 f_6 a_2 a_1^2 - f_6 a_1^4. \tag{3.9}$$

One can check directly that x_1^3 and x_2^3 are the highest powers of x_1 and x_2 in the numerator of the right hand side, so that $o_n(\text{numerator}) = 6$ and $o_n(\kappa_3) = 2$, as required.

For the 16 basis elements of $L(4\Theta)$, one cannot simply apply the above process to monomials in a_0, a_1, a_2, b_0, b_1, and it is necessary to consider more general polynomials. For example,

$$z_7 = -a_2 b_0 - a_1 b_1. \tag{3.10}$$

Similarly, all of $z_1, \ldots, z_{15}$ in (2.10) can be written as polynomials in a_0, a_1, a_2, b_0, b_1.

3.8 $L(2\Theta)$ in Genus 3

In genus 3, we find that (3.1),(3.2),(3.3) are

$$
\begin{gathered}
a_0 = 1,\ a_1 = x_1 + x_2 + x_3, \\
a_2 = x_1x_2 + x_2x_3 + x_3x_1,\ a_3 = x_1x_2x_3, \\
b_0 = (x_1y_2 - x_2y_1 - x_3y_2 + x_3y_1 - x_1y_3 + x_2y_3)/\delta, \\
b_1 = (x_3^2y_2 - x_3^2y_1 + x_2^2y_1 + y_3x_1^2 - y_2x_1^2 - y_3x_2^2)/\delta, \\
b_2 = (-x_2^2x_3y_1 + x_2^2x_1y_3 + x_3^2x_2y_1 - y_3x_2x_1^2 + y_2x_3x_1^2 - x_3^2x_1y_2)/\delta, \\
\delta = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1).
\end{gathered}
\tag{3.11}
$$

In genus 3, we find that, even for $L(2\Theta)$ it is not enough merely to consider monomials in the a_i, b_j. There is only one case, namely that of b_0^2 for which the computations are similar to b_0^2 in genus 2. Namely, one adjusts the numerator modulo δ^2 to obtain a member of $L(2\Theta)$ in the form b_0^2 added to a polynomial in $a_0, \ldots, a_3$.

$$
c_0 = a_0b_0^2 - f_7a_1^3 + f_7a_2a_1 - f_6a_1^2 + 3f_7a_3 + 2f_6a_2. \tag{3.12}
$$

However, we know from Lemma 2.5 that the dimension of $L(2\Theta)$ is $2^g = 8$, and a_0, a_1, a_2, a_3, c_0 are five independent elements. We are therefore missing three further independent elements to make up a basis, and hence a set of coordinates for the Kummer variety. We search amongst linear combinations of monomials of the form $a_ib_jb_k$ to obtain expressions that can be weight reduced to give new independent members of $L(2\Theta)$. The expressions quickly become large, and so it is helpful to have the following technical lemma.

Lemma 3.12 *Let wt be a weight for which $\mathcal{C} : y^2 = f_n x^n + \ldots + f_0$ and δ are homogeneous. Let c be a member of $L(n\Theta)$ with denominator a power of δ. Let $c_{i_1}, \ldots, c_{i_m}$ be the homogeneous pieces of c satisfying $c = c_{i_1} + \ldots + c_{i_m}$ and $wt(c_{i_1}) = i_1, \ldots, wt(c_{i_m}) = i_m$, where $i_1, \ldots, i_m \in \mathbb{Z}$ are distinct. Then each of $c_{i_1}, \ldots, c_{i_m}$ is a member of $L(n\Theta)$.*

Proof Let t be any scalar; that is, t is a member of the groundfield K which, as always, we assume to be of characteristic 0. Replace every occurrence of x, y, x_j, y_k, f_ℓ by $t^{wt(x)}x, t^{wt(y)}y, t^{wt(x)}x_j, t^{wt(y)}y_k, t^{wt(f_\ell)}f_\ell$, respectively. Then the curve $\mathcal{C}$ is unchanged, and so the corresponding $c(t)$ will still be in $L(n\Theta)$ for any value of t. But $c(t) = t^{i_1}c_{i_1} + \ldots + t^{i_m}c_{i_m}$. We can now use the inverse of the invertible matrix $(j^{i_k})_{jk}$ to express each of $c_{i_1}, \ldots, c_{i_m}$ as a linear combination of $c(1), \ldots, c(m)$. Hence, each $c_{i_1}, \ldots, c_{i_m}$ is in $L(n\Theta)$, since membership of $L(n\Theta)$ is preserved by linear combinations. $\square$

The above lemma provides an invaluable computational aid, since it tells us that there exists a basis for $L(n\Theta)$, where each basis element is homogeneous with respect to any weight satisfying the conditions of the lemma. Note that these conditions are not met by o_n itself, but they are satisfied, for example, by the x and y weights in Figure 3.1. This means that when we search for linear combinations of monomials, we can first partition them into sets of equal x and y weights.

Taking the various squares and cross-products we notice that the coefficients of $y_i y_j$ in the numerator are still too 'heavy' compared to the denominator, and so we try to find linear combinations of $a_i b_j b_k$ which have a $y_1 y_2$ term (in the numerator) of weight 10 or less; then by symmetry we can concentrate solely on reducing the weight of the rest of the terms in these linear combinations.

The process is made more efficient by, for each product, considering the $y_1 y_2$ coefficient, separating the heaviest portion (in this case weight> 10) and symmetrising[1] this portion[2]. Then obtaining a basis for the null space of the set of the symmetric heavy portions gives the required linear combinations of $a_i b_j b_k$ which have acceptably heavy (or 'light') $y_i y_j$ terms.

[1]Not as trivial as it sounds; implemented in `gpolyutil[get_orbit]()`

[2]Separation & re-symmetrification performed by `gpolyutil[get_weight_greaterthan]()`

We find that the required linear combinations of the $a_i b_j b_k$ are

$$\begin{aligned}
\gamma_0 &= a_0 b_0^2 \\
\gamma_1 &= a_1 b_0^2 + 2a_0 b_0 b_1 \\
\gamma_2 &= a_0 b_1^2 - a_2 b_0^2 \\
\gamma_3 &= a_1 b_1^2 + 2a_2 b_0 b_1 + a_3 b_0^2
\end{aligned} \tag{3.13}$$

Note however that the γ_i are still too heavy, but after weight reduction[3] we find wred $\gamma_i \in L(2\Theta)$ as required.

Using these ideas, one can determine the $\mathcal{L}(2\Theta_g)$ embedding for a hyperelliptic curve of genus $g = 3$.

Theorem 3.13 *Let $a_0, a_1, a_2, a_3, b_0, b_1, b_2$ be as in (3.11). A basis for $L(2\Theta)$, and hence an embedding in $\mathbb{P}^8$ of the Kummer variety, is given by:*

$$[\kappa_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5, \kappa_6, \kappa_7] = [a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3]$$

where

$$\begin{aligned}
c_0 &= a_0 b_0^2 - f_7 a_1^3 + f_7 a_2 a_1 - f_6 a_1^2 + 3 f_7 a_3 + 2 f_6 a_2 \\
c_1 &= a_1 b_0^2 + 2a_0 b_0 b_1 - f_7 a_1^4 + 3 f_7 a_2 a_1^2 - f_6 a_1^3 - f_7 a_2^2 \\
&\quad - f_7 a_3 a_1 + 2 f_6 a_2 a_1 - f_5 a_1^2 + 2 f_5 a_2 \\
c_2 &= a_0 b_1^2 - a_2 b_0^2 + f_7 a_2 a_1^3 - 2 f_7 a_2^2 a_1 + f_6 a_2 a_1^2 \\
&\quad + f_7 a_3 a_2 - f_6 a_2^2 + f_5 a_2 a_1 - 3 f_5 a_3 \\
c_3 &= a_1 b_1^2 + 2a_2 b_0 b_1 + a_3 b_0^2 \\
&\quad + f_7 a_2^2 a_1^2 - f_7 a_3 a_1^3 + f_7 a_3 a_2 a_1 - f_7 a_2^3 + f_6 a_2^2 a_1 \\
&\quad - f_6 a_3 a_1^2 + f_5 a_2^2 - f_5 a_3 a_1.
\end{aligned}$$

Proof We know from Lemma 2.5 that the dimension of $L(2\Theta)$ is 8. Furthermore it may be directly checked algebraically that the above are all linearly independent, for example by examining the coefficient of $y_i y_j$ in the numerators of c_0, c_1, c_2, c_3,

[3] `gpolyutil[naive_reduce_g3()]`

and so $a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3$ are a basis. It is sufficient to demonstrate the independence of the c_i for $i = 0, \ldots, 3$. Considering the coefficient of $y_1 y_2 (x_1 - x_3)(x_2 - x_3)$ in the numerator of c_i gives

$$\begin{aligned} c_0 &\longmapsto -2 \\ c_1 &\longmapsto 2x_3 \\ c_2 &\longmapsto -2x_3^2 \\ c_3 &\longmapsto -2x_3^3 \end{aligned} \tag{3.14}$$

which shows that the c_i are independent as required. □

We have deliberately avoided mention of the case where the genus is odd and $y^2 = F(x)$, where the degree of $F(x)$ is even. In genus 3, this means that we have considered only $F(x)$ of degree 7. However, it is worth noting that one can formally apply precisely the same procedures as above to the case where $F(x)$ has degree 8. In that case, $o_n(x) = 2$, $o_n(y) = 8$, and we can take the weight reductions of precisely the same expressions as in (3.13). The resulting revised versions of $c_0, \ldots c_4$ (which will include terms with a coefficient of f_8) still have only $o_n = 2$. The problem here (if one wants to retain objects defined over the ground field) is giving an interpretation of the x_1, x_2, x_3, y_1, y_2, y_3 in terms of divisor classes, since each divisor class is no longer canonically represented by three points. The problem in representing divisor classes by sets of points is that there is no equivalent of $g\Theta$ (when the degree of $F(x)$ is odd) or $(g/2)\Theta^+ + (g/2)\Theta^-$ (when the genus is even). We can pick one of Θ^+ or Θ^-, but neither is defined over the ground field. However, as -1 acts trivially on the Kummer variety and $c_0, \ldots, c_3$ furnish an embedding of this, it may be possible to extend an interpretation of $c_0, \ldots, c_3$ to the degree 8 case, although we have not been able to do so. Note that in genus 4 there would be no such problems, since the genus is even, and so the above techniques (like in genus 2) would apply equally well to the cases where $F(x)$ has even or odd degree.

3.9 Calculating the c_i

The c_i may be generated easily using the software in the `hell` subdirectory of the ftp site. To read in to `maple` all of the files necessary, use `thesis.v5`; then the c_i may be deduced from the output of the function

$$\texttt{get\_nTheta(genus,wWeight,yWeight,nTheta)} \tag{3.15}$$

as follows:

$$\begin{aligned}
\texttt{get_nTheta(3,4,2,2)} &\longmapsto c_0 \\
\texttt{get_nTheta(3,5,2,2)} &\longmapsto c_1 \\
\texttt{get_nTheta(3,6,2,2)} &\longmapsto c_2 \\
\texttt{get_nTheta(3,7,2,2)} &\longmapsto c_3.
\end{aligned} \tag{3.16}$$

The function returns an array, which maps order of the pole on Θ to the function; hence, the appropriate functions are returned in the `[2]` element. Also, note that this is a list of polynomials in the a_i, b_i, from which an appropriate can be chosen. Further, note that the function uses a triple pass of the process of distillation (3.6) which returns results which are *formally valid*, in the sense that they are simply polynomials that match the weight parameters given without regard to whether the results make sense (particularly with respect to the odd genus/even degree problem described above); hence, the results will contain references to f_8, although if the assignment $f_8 = 0$ is subsequently made, the results can be chosen from the lists offered to be identical to the ones given in (3.13).

3.10 $L(3\Theta)$ in Genus 3

For $L(3\Theta)$ in genus 3, we know that the dimension is 27. Furthermore, a_0, a_1, a_2, a_3, b_0, b_1, b_2 of (3.11) are all in $L(3\Theta)$, as are the c_0, c_1, c_2, c_3 of Theorem 3.13. This gives 11 independent members of $L(3\Theta)$, and so 16 additional independent members are required to give a basis, and hence an embedding of the Jacobian as a projective variety. Our strategy for finding these is the same as for finding c_0, c_1, c_2, c_3; however, problems with the sheer size of the algebra involved has prevented the explicit determination of an entire basis. Nevertheless, we have made some progress towards a basis, as described in the following partial result.

Theorem 3.14 *Let $a_0, a_1, a_2, a_3, b_0, b_1, b_2$ be as in (3.11), and c_0, c_1, c_2, c_3 be as in Theorem 3.13. Then these, together with the following, give 15 independent members of the 27-dimensional space $L(3\Theta)$.*

$$\begin{aligned}
d_0 &= a_1{b_0}^2 + a_0b_0b_1 \\
&\quad + 2\,f_5{a_0}^3a_2 - f_5{a_0}^2{a_1}^2 + 3\,f_6{a_0}^3a_3 + f_6{a_0}^2a_1a_2 \\
&\quad - f_6a_0{a_1}^3 - f_7{a_0}^2{a_2}^2 + 2\,f_7a_0{a_1}^2a_2 - f_7{a_1}^4, \\
d_1 &= a_0b_0b_2 - f_6a_0^2a_1a_3 - f_7a_0a_1^2a_3, \\
d_2 &= a_0b_1b_2 + a_3b_0^2 \\
&\quad - f_5{a_0}^2a_1a_3 + f_6{a_0}^2a_2a_3 - f_6a_0{a_1}^2a_3 + 2\,f_7a_0a_1a_2a_3 - f_7{a_1}^3a_3, \\
d_3 &= a_1b_1b_2 + a_2b_0b_2 - a_3b_0b_1 \\
&\quad - f_5{a_0}^2a_2a_3 - f_6a_0a_1a_2a_3 + f_7a_0{a_2}^2a_3 - f_7{a_1}^2a_2a_3.
\end{aligned}$$

Proof It is sufficient to demonstrate the independence of the c_i, d_i for $i = 0, \dots, 3$. Considering the coefficient of $y_1y_2(x_1 - x_3)(x_2 - x_3)$ in the numerator of the c_i, d_i extends (3.14) as follows.

$$\begin{aligned}
c_0 &\longmapsto -2 \\
c_1 &\longmapsto 2x_3 \\
c_2 &\longmapsto -2x_3^2 \\
c_3 &\longmapsto -2x_3^3 \\
d_0 &\longmapsto -(x_1 + x_2) \\
d_1 &\longmapsto -(x_1 + x_2)x_3 \\
d_2 &\longmapsto (x_1 + x_2)x_3^2 \\
d_3 &\longmapsto (x_1 + x_2)x_3^3
\end{aligned} \tag{3.17}$$

which shows that the c_i, d_i are independent as required. □

Note that, by inspection, the c_i, d_i are homogeneous in the x, y and w weights of Figure 3.1. We can take the case $g = 3$, $n = 7$ of Figure 3.1 and extend the weights to the c_i of (3.13) and d_i of (3.14), as follows.

We presume that to find the missing 12 members of the basis of $L(3\Theta)$ would require a search through monomials cubic in the b_i. At the moment there are

	x	y	w	o_7
x_i	1	0	1	2
y_i	0	1	4	7
f_i	$-i$	2	$8-i$	0
a_i	i	0	i	2
a_4	0	1	4	7
a_5	1	1	5	9
a_6	2	1	6	9
b_i	$i-2$	1	$2+i$	3
c_i	$i-4$	2	$4+i$	2
d_i	$i-3$	2	$5+i$	3

Figure 3.4: Weights and Measures

simply too many of these to perform an exhaustive search, even with the enhancements described above.

Chapter 4

Algorithms for Determining $\mathcal{L}(n\Theta)$

The aim of this chapter is to give detail and precision to the descriptions of the algorithms motivated by the previous sections.

4.1 Finding Monomials with a Given Weight

To determine the list of the monomials of combinations of the symbols f_i, a_i, b_i, at a particular w-weight, w, and y-weight, y, we require a *weighted partition* of w and y. For this purpose, we need a list of weighted counters, $[c_1, \dots, c_n]$, such that there is a $1:1$ correspondence between the c_i and the f_i, a_i and b_i; and each c_i has w-weight w_i and y-weight y_i. Hence the only permissible values of the counters are those with $\sum c_i w_i = w$, and $\sum c_i y_i = y$. As it is possible that $y_j = 0$ for some specific j, it is simpler (and far quicker) to perform both comparisons at the same time.

In this implementation of the function `wpartition`, the expected arguments are in pairs; the first in the pair being the maximum allowed weight sum for that measure, and the second being a list of weights for the counters (which also gives the number of counters required): e.g., to partition the number 3 into 2 parts, each with weight 1: `wpartition(3,[1,1])` returns

$$[[3,0],[2,1],[1,2],[0,3]]$$

and to partition the number 3 into 2 parts with the second part being weighted at 2: `wpartition(3,[1,2])` returns

$$[[3,0],[1,1]]$$

```
andy_util[ wpartition ] := proc()
  local flag, ctr_idx, ctr_cur, ctr_new, ctr_len, retval;

  if (nargs<2) then ERROR(`too few arguments`) fi;
  if type(nargs,odd) then ERROR(`wrong number of arguments`) fi;

  # retval  the value we return; this will be a list of a copy
  #         of the counter list each time it meets the weight
  #         criteria
  # args[2] must be a list of weights; the list will contain the
  #         same number of elements as the number of counters we
  #         require so we init the current counter, ctr_cur with
  #         zeros exactly to the right length.
  # flag    will be set to true upon a counter overflow; at which
  #         point we have ticked past the end of the counter
  #         through all the possibilities, and may return.
  retval:=[];
  ctr_len:=nops(args[2]);
  ctr_cur:=[0$ctr_len];
  flag:=false;

  while(flag=false) do
    # the function checkweights verifies whether the current
    # counter meets all the criteria specified; the 'e'
    # argument refers to 'equal to'; as the function can also
    # check for 'le', i.e., 'less than or equal to'.
    # If the criteria are met, it returns true, in which case
```

```
    # the value of the current counters is appended to the
    # return value.
    if andy_util[checkweights](ctr_cur,"e",args)
      then retval:=[op(retval),ctr_cur] fi;
    # the ctr_inc function increments the current counter list
    # on a 'tick'; so that if incrementing counter c_i causes
    # the weight to be too high for some weight specification,
    # then c_1 to c_i are all reset to zero, and c_i+1 is
    # incremented. If i+1 is past the end of the list, then
    # we have overflowed and wpartition can return.
    # ctr_inc returns a list; [boolean,new_counter] where
    # the boolean refers to whether or not an overflow occurred.
    andy_util[ctr_inc](ctr_cur,args);
    flag:=%[1]; ctr_cur:=%%[2];
  od:

  retval;
end:
```

The function ctr_inc is implemented as follows

```
andy_util[ ctr_inc ] := proc( ctr::list )
  local ctr_len, ctr_idx, ctr_new, flag;

  # the arguments following the list provide the same information
  # as given to wpartition above

  ctr_len:=nops(ctr);
  # try to inc the least significant counter first
  ctr_idx:=1;
  # assume that the counter will overflow on this inc
  # this flag also doubles as being the logical inverse of
  # whether the counter has been incremented successfully
  flag:=true;
```

```
  # the condition below translates to:
  #   whilst we have not incremented successfully, and have not
  #   overflowed...
  while(flag=true) and (ctr_idx<=ctr_len) do
    # form a new list in which the counters up to ctr_idx are
    # all nullified, and the counter at ctr_idx is incremented
    # by 1.
    ctr_new:=[ 0$(ctr_idx-1), ctr[ctr_idx]+1,
               op(ctr[ctr_idx+1..ctr_len])];
    # if the increment is still too heavy, then we will need to
    # repeat the operation again on the next counter
    ctr_idx:=ctr_idx+1;
    if(andy_util[checkweights](ctr_new,"le",args[2..nargs]))
    then flag:=false fi;
  od:
  # at this point the flag is false only if we managed to
  # increment the counter list with a weight within the
  # boundaries specified in the arguments; otherwise there
  # was an overflow and there are no more values possible
  [flag,ctr_new];
end:
```

The function `checkweights` is used by both `wpartition`, which requires a test for equality, and `ctr_inc`, which requires a test for less than or equal to, and is implemented as follows:

```
andy_util[ checkweights ] := proc( check::list, rule::string )
  local i, value, weightlist;

  # the first two supplied arguments are the list that requires
  # checking, and the rulestring, which determines whether the
  # test is for equality, or for ‘less than or equals’.
  # the following arguments are as supplied to wpartition; i.e.,
  # total weight, and a list of each component weight
```

```
  for i from 3 to nargs by 2 do
    # we cycle through all the weight/[component weight] values
    # testing each time
    value:=args[i];
    weightlist:=args[i+1];
    # n.b. there is a small efficiency question here, as these
    # tests are made on every loop; it would be more efficient,
    # although programmatically more messy to perform the test
    # once and have two different loops
    if ((rule="e") and
        (value<>sum('weightlist[i]*check[i]',
                    'i'=1..nops(check))))
      then RETURN(false) fi;
    if ((rule="le") and
        (value<sum('weightlist[i]*check[i]','i'=1..nops(check))))
      then RETURN(false) fi;
  od;
  # we return true iff every test passed successfully
  true;
end:
```

Finally, to obtain the list of monomials in the symbols f_i, a_i, and b_i, with w-weight w and y-weight y, we use the function `get_monos`; e.g., `get_monos(8,4,2)`, which will return a list of monomials of w-weight 4 and y-weight 2 on a hyperelliptic curve defined by an equation of degree 8 in x. The function is:

```
hell2[ get_monos ];=
     proc( deg::integer, wweight::integer, yweight::integer )
  local genus, fablist, fabwwt, fabywt_hyper, i, j, exps;

  genus:=floor((deg-1)/2);
  # create a list of the f_i, a_i, and b_i; this will correspond
  # to the counters used
  fablist:=[ seq(f[i],i=0..deg),
```

```
            seq(a[i],i=1..2*genus),
            seq(b[i],i=0..genus-1)];
  # the w-weight of each of the symbols in the above list is
  # placed in a list
  fabwwt:=[seq(hell2[wwt](deg,fablist[i]),i=1..nops(fablist))];
  # the y-weight of each of the symbols in fablist is now
  # placed in a list
  fabywt_hyper:=
    [seq(hell2[ywt_hyper](deg,fablist[i]),i=1..nops(fablist))];

  # now calculate the weighted partition of the w-weight and
  # y-weight given corresponding weights of the counters
  # exps will be a list of weighted partitions satisfying the
  # stated constraints
  exps:=
  andy_util[wpartition]( wweight, fabwwt,
                         yweight, fabywt_hyper );

  # for each partition in exp, create the monomial, and add it
  # to a list, which becomes the return value of this function
  [seq(product('fablist[i]^exps[j][i]',
                  'i'=1..nops(fablist)),j=1..nops(exps))];
end:
```

4.2 Weight Reduction

As discussed earlier, to each symbol can be associated a hyperelliptic weight. In particular, for a genus g curve $\mathcal{C}$ defined by a hyperelliptic equation of degree d in x, we can associate a weight, 2, with x, and a weight d with y. The function `g3weight` returns the hyperelliptic weight of the symbols x, and y for a genus 3 curve given by a degree 7 equation $F(x, y) = 0$ with $F(x, y) = y^2 - f_0 + \cdots + f_7 x^7$. Note that F is pseudo-homogeneous with respect to this weight.

Definition 4.1 *Suppose p is a rational symmetric function on $\mathcal{C}^3$, defined in terms of (x_i, y_i), for $1 \leq i \leq 3$. Then the* `g3weight` *of this function is defined by the weight of its denominator subtracted from the weight of its numerator, the weights of which are determined by calculating the degree of p wrt w when (x_1, y_1) is substituted by (wx_1, wy_1).*

Note that the use of x_1, y_1 as a 'pivot' is a common technique in these algorithms.

```
g3w(p((x1,y1),(x2,y2),(x3,y3)))=
       degree(p((w*x1,w*y1),(x2,y2),(x3,y3)),w)
g3weight(p)=g3w(numerator(p))-g3w(denominator(p))
```

Often in the process of 'weight reduction' we wish to obtain the heaviest portion, h, of an expression p. Again, we pivot on x_1, y_1 and construct an expression e by considering the weight of each term t in p. If t is heavier than e then we discard the contents of e and start again by setting `e:=t`; on the other hand, if t is the same weight as e then we append t to e, thus: `e:=e+t`. Otherwise, no action is needed. The return from this routine, shown below, is no longer symmetric, as just terms in x_1, y_1 are considered, but a symmetric result can be obtained by using another function, `get_orbit`.

```
get_heaviest_terms:=proc(p,g)
  local m, n, terms, poly;
  poly:=expand(x[1]*y[1]*p):
  terms:=0:
  for m from 1 to degree(poly,y[1]) do
    expand(coeff(collect(poly,y[1]),y[1]^m));
    n:=degree(%,x[1]);
    if n>0 then
      coeff(collect(expand(%%),x[1]),x[1]^n);
      if gpolyutil[g3weight](terms)=((2*g+1)*(m-1)+2*(n-1))
        then terms:=terms+x[1]^(n-1)*y[1]^(m-1)*% fi;
```

```
    if gpolyutil[g3weight](terms)<((2*g+1)*(m-1)+2*(n-1))
      then terms:=x[1]^(n-1)*y[1]^(m-1)*% fi;
  fi;
od:
expand(terms);
end:
```

Before get_orbit(), we require a simple function permute_vars(p,sigma) that permutes the (x_i, y_i). Essentially this function works by substituting x_i with t_i, and y_i with s_i, before then substituting t_i with $x_{i\sigma}$, and s_i with $y_{i\sigma}$. Exactly how σ is represented is implementation dependent. For instance, in Maple V, it is useful to use the variable length parameter capability, so that the number in the $(i+1)$-th position in the parameter list represents the j in the x_j which x_i will become.

Consider the operation strip, which strips from an expression all terms which contain neither x_1 nor y_1. Then, given that the original expression was symmetric in the (x_i, y_i), the resultant expression still contains enough information to reconstruct the original: this reverse operation is the purpose of the function get_orbit(poly,genus). In essence, this function obtains a list of permutations of the terms in p and, after discarding any repetitions, returns the sum of all the terms in that list. For instance,

```
get_orbit(x1+x2+x1*x2,3) = x1+x2+x3+x1*x2+x1*x3+x2*x3
```

```
gpolyutil[get_orbit]:=proc(poly,genus)
  local perms, terms, newterms, i, j;
  perms:=combinat[permute](genus):
  if type(expand(poly),`+`)
    then terms:=[op(expand(poly))]
    else terms:=[poly] fi;
  newterms:=;
  for i from 1 to nops(terms) do
    newterms:= op(newterms),
                seq(gpolyutil[permute_vars](terms[i],
```

```
                                              op(perms[j])),
                      j=1..nops(perms));
  od;
  sum('newterms[i]','i'=1..nops(newterms));
end:
```

Often we are interested not just in terms of a particular weight, but in all terms heavier than some specific weight. Clearly we need another function, `get_weight_greaterthan`. This takes 3 arguments, specifying the desired weight, the expression to be 'skimmed', and the genus of the underlying curve. The function simply needs to call `get_heaviest_terms` repeatedly, making a note of which terms are heaviest, removing them, and finally returning the heaviest terms in a single expression.

```
gpolyutil[get_weight_greaterthan]:=proc(wght::integer,
                                         p::polynom,
                                         genus::integer)
  local symp, tempval, retval;
  if (genus <> 3)
    then ERROR(`only genus 3 implemented - weight function`) fi;
  symp:=get_orbit(p,genus);
  tempval := 0;
  retval := 0;
  while (g3weight(symp) > wght) do
    tempval := get_orbit(get_heaviest_terms(symp,genus),genus);
    retval := retval + tempval;
    symp := simplify(symp-tempval);
  od;
  retval;
end:
```

4.3 Distillation

The key distillation algorithms, as used in section 3.9, are described here. The first one, `distillation` takes 2 arguments; the degree of F, and a list of monomials. The algorithm depends on tables of lists of the monomials in different formats (both expanded into the x_i, y_i representation and the smaller a_i, b_i representation.

```
# uses a lot of Maple V R5 lexical scoping
hell[ distillation ] := proc( deg::integer, monos::list )
  local init_distil, next_distil_level, get_top_portion,
w_subs_hyper, process_lists, update_tables,
distil_levels, w, ndl, emweight,
        sym_table, exp_table, exp_list, top_list, top_kbasis,
        orders, genus, i, m, mweight, wweight;

  #######################################################
  #### subprocedures

  # creates table of lists of monomials indexed by their order
  init_distil := proc()
    local m, morder;
    sym_table:=table([]);
    for m in monos do
      if type(m,`+`)
        then morder := hell[order_hyper]( deg, m );
        else morder := hell[order_symbol]( deg, m );
      fi;
      if not member( [morder], [indices(sym_table)] )
      then sym_table[morder]:=[]; exp_table[morder]:=[]
      fi;
      sym_table[morder]:=[op(sym_table[morder]),m];
      exp_table[morder]:=[op(exp_table[morder]),infinity];
    od:
    distil_levels := indices(sym_table);
```

```
end:

next_distil_level := proc()
  local i, ndl;
  if (nops(distil_levels) = 0) then RETURN(-infinity) fi;
  jdl := max(seq(op(distil_levels[i]),
                 i=1..nops(distil_levels)));
  distil_levels := distil_levels minus [ndl];
  ndl;
end:

w_subs_hyper := proc( expr )
  local xyexpr;
  xyexpr := hell[subs_hyper]( deg, expr );
  subs( x[1]=w^orders[x]*x[1], y[1]=w^orders[y]*y[1], xyexpr );
  collect( expand(w*%), w );
end:

process_lists := proc()
  local new_exp_list, i;
  new_exp_list := [];
  for i from 1 to nops(exp_list) do
    if (exp_list[i]<>infinity)
    then new_exp_list := [op(new_exp_list),exp_list[i]]
    else new_exp_list :=
      [op(new_exp_list),
       w_subs_hyper(delta^emweight*sym_table[ndl][i])]
    fi;
  od;
  exp_list := new_exp_list;
  top_list :=
    [seq(coeff(exp_list[i],w^(ndl+1+emweight*orders[delta])),
         i=1..nops(exp_list))];
end:
```

```
update_tables := proc()
  local i, basis_elt, ab_expr, wxy_expr, wxy_ord;
  for basis_elt in top_kbasis do
    ab_expr  := sum('sym_table[ndl][i]*basis_elt[i]',
                    'i'=1..nops(exp_list));
    wxy_expr := sum('exp_list[i]*basis_elt[i]',
                    'i'=1..nops(exp_list));
    wxy_ord  := degree(collect(expand(wxy_expr),w),w)
                -1-emweight*orders[delta];
    if not ( member([wxy_ord], [indices(sym_table)]) )
    then
      distil_levels:=op(distil_levels),[wxy_ord];
      sym_table[wxy_ord]:=[];
      exp_table[wxy_ord]:=[]
    fi;
    sym_table[wxy_ord]:=[op(sym_table[wxy_ord]),ab_expr];
    exp_table[wxy_ord]:=[op(exp_table[wxy_ord]),wxy_expr];
  od;
  exp_table[ndl]:=[]; exp_list:=[]; # free up some space !!!!
end:

########################################################
#### main
# this is the entry point of the algorithm.
wweight := hell[wwt](deg, monos[1]);
mweight := hell[mwt](deg, monos[1]);
#  monos := hell[get_monos]( deg, wweight, mweight );

#  monos := [seq(monos[i]*delta^mweight,i=1..nops(monos))];


emweight := 2*ceil(mweight/2);
```

```
  genus := floor((deg-1)/2);
  orders[x]     := hell[order_hyper]( deg, x[1] );
  orders[y]     := hell[order_hyper]( deg, y[1] );
  orders[delta] := hell[order_hyper]( deg, delta );

  init_distil();

  ndl := next_distil_level();
  while(ndl<>-infinity) do
    printf("%d ",ndl);
    exp_list := exp_table[ndl];
    process_lists();
    top_kbasis := mvutil[get_null_space](top_list,
                                        [seq(x[i],i=1..genus),
                                         seq(y[i],i=1..genus)]);
    update_tables();
    ndl := next_distil_level();
  od;

  eval(sym_table);
end:
```

The following algorithm uses the `distillation` function to perform a triple-pass distillation in order to deal with both cases, F being of odd or even degree, and return a list of monomials of the correct weight, and having the correct order on Θ (when this makes sense; cf 3.9).

```
hell[ get_nTheta ] := proc( genus::integer, wweight::integer,
                            mweight::integer, nTheta::integer)
  local odeg, edeg, i, j, m,
        emonos, eptable, omonos, optable, eolist,
        eTheta, oTheta, mature, theta_table, varlist;

  # given the genus, calculate both odd and even degrees
```

```
eTheta  := 2*ceil(nTheta/2);
oTheta  := eTheta-1;
edeg    := 2*genus+2;
odeg    := edeg-1;

# first distillation is for even degree
emonos  := hell[get_monos](edeg, wweight, mweight);
eptable := hell[distillation](edeg, emonos);

# second distillation is for odd degree
omonos  := hell[get_monos](odeg, wweight, mweight);
optable := hell[distillation](odeg, omonos);

# mix the tables; fist add the polynomials from the odd
# degree distillation
eolist  := [];
for i in indices(optable) do
  if (op(i) <= eTheta)
    then eolist:=[op(eolist),op(optable[op(i)])];
  fi;
od;

# next, add those polynomials from the even degree
# distillation whose coefficient of f[edeg] is not 0
for i in indices(eptable) do
  for j from 1 to nops(eptable[op(i)]) do
    if coeff(collect(eptable[op(i)][j],f[edeg]),f[edeg])<>0
      then eolist := [op(eolist),eptable[op(i)][j]];
    fi;
  od;
od;

# final pass on distillation
mature := hell[distillation](edeg,eolist);
```

```
  theta_table := table([eTheta=[],oTheta=[]]);
  for m in mature[eTheta] do
    i := hell[order_hyper](odeg,m);
    if i<>(-infinity) and (hell[bwt](edeg,m)=mweight)
      then theta_table[i]:=[op(theta_table[i]),m];
    fi;
  od;

  varlist := [seq(a[i],i=1..2*genus),seq(b[i],i=0..genus-1),
              seq(f[i],i=0..edeg)];

  table([eTheta=mvutil[get_basis](theta_table[eTheta],varlist),
 oTheta=mvutil[get_basis](theta_table[oTheta],varlist)]);
end:
```

4.4 Overview of the Implementation of Distillation

The implementation of the distillation algorithm, `distillation`, given above, makes a lot of use of a feature of Maple V introduced in release 5; namely *lexical scoping*. This essentially allows procedures defined within other procedures access to the local variables used by the parent procedure. As this can be quite confusing, this section lists, in the style of technical documentation, the various variables and sub-procedures and how they interact. This will serve to allow for an example of the implementation at work.

4.4.1 Variables

`w`	used as a formal variable
`mweight`	used only to calculate `emweight`
`emweight`	minimum power δ must be multiplied by to generate a list of symmetric polys, as opposed to symmetric ratpolys, which may possibly have antisymmetric numerator and denominator.
`genus`	the genus of the curve
`orders`	a list of orders of symbols, indexed by the symbol; this is just an optimisation
`sym_table`	table of lists of expressions in f,a,b indexed by its level; i.e., o, the order of the pole at ∞ of that expression
`exp_table`	copy of `sym_table` with each expression multiplied by δ^{χ}, where χ = `emweight`, and expressed in terms of f,x,y,w where the power of w in any term is equal to 1 plus the order of the pole at ∞ of its coefficient. Expanding a given expression is deferred until the expression is required; until then, its place is marked with the symbol ∞. When a level has been completed, the expanded expressions for that level are no longer required.
`ndl`	the index into `sym_table` for the main loop of the algorithm; the current level that is being processed.
`exp_list`	the fully expanded list from `exp_table` corresponding to `ndl`
`top_list`	a list corresponding to `exp_list`, but with each entry truncated to include only the coefficient of the leading term of `w`.
`top_kbasis`	the set of vectors which form the basis for the null space of the polynomials in `top_list`
`distil_levels`	the set of levels yet to be processed

4.4.2 Subprocedures

init_distil	
Initialize the main data structures for the algorithm	
Variables used	
`monos`	read-only
`sym_table`	initialize tableof lists of expressions indexed by level
`exp_table`	initialize table by placing marker in every position
`distil_levels`	initialize with the keys from `sym_table`

next_distil_level	
Determine next level to process. This function returns the highest entry from `distil_levels`, or $-\infty$ if empty.	
Variables used	
`distil_levels`	the return value is removed from the list

w_subs_hyper	
Given an expression, `expr`, in terms of a, b, expand into expressions in terms of x, y, and introduce a formal variable, w, such that the coefficient of the term w^{n+1} is a symmetric polynomial in the pairs (x_i, y_i) with pole of order n at ∞. The expanded expression is the return value.	
Parameters	
`expr`	the expression to expand

`process_lists`	
Process `exp_list`, replacing any ∞ markers with expanded expression, muliplying each expression by δ^{χ} (where $\chi =$ `emweight`) to remove all denominators, and ensure all expressions are symmetric, rather than anti-symmetric, in the pairs (x_i, y_i). N.B., that these expressions will be polynomials in the variable w. The variable `top_list` is generated by considering the coefficients of the leading terms in w of each of the members of `exp_list`.	
Variables used	
`emweight`	read-only
`exp_list`	list of expanded entries
`top_list`	list of coefficients of leading term in w in corresponding expressions in the `exp_list`.

`update_tables`	
The variable `top_kbasis` contains a set of vectors which give a basis such that, when applied to the members of `sym_table` (and `expr_table`), give expressions the order of whose poles at ∞ are strictly less than those of the original expressions. The new expressions are added to the tables `sym_table` and `exp_table`, and `distil_levels` is updated if necessary.	
Variables used	
`top_kbasis`	vector list containing basis for null space of `top_list`.
`sym_table`	new entries are added, as determined by `top_kbasis`.
`exp_table`	new entries are added, in a similar fashion to `sym_table`; old entries which are not needed again are discarded.
`exp_list`	used, along with `top_kbasis` to generate new entries in `exp_table`; its value is discarded before return.
`distil_levels`	as a result of adding new entries to the tables, it might be necessary to process more levels; they are added if necessary.

4.5 Example Distillation

As an example of the behaviour of my implementation of the distillation algorithm, we now consider how it calculates c_1 in genus 3, as given in Theorem 3.13. This is a good representative of a nontrivial distillation in which the expressions are still vaguely manageable by hand.

We know that c_1 has w-weight 5, and y-weight 2, but if we had not known these values, we could have arrived at them by deciding to fix on a low value for the y-weight, which essentially limits the highest power of the b_i's in the expressions, and then searching through the w-weights from the minimum possible for the given y-weight.

Choosing the y-weight to be 2 limits the expressions to be quadratic in the b_i's, and, given that the lightest (in terms of the w-weight) expression that contains a quadratic b_i is b_0^2, which has weight 4, this is a sensible place to start. Choosing 4 gives c_0; here we choose 5 to obtain c_1.

We use the function `get_monos(7,5,2)` to obtain a comprehensive list of suitable monomials in f,a,b:

$$f_3, f_4a_1, a_1^2f_5, a_1^3f_6, a_1^4f_7, f_5a_2, a_1a_2f_6, a_1^2a_2f_7, a_2^2f_7, f_6a_3, a_1a_3f_7, a_1b_0^2, b_0b_1. \quad (4.1)$$

Next, we call the distillation algorithm using this list of monomials; we can combine these two steps into one by using the following:

```
distillation(7,get_monos(7,5,2));
```

The distillation process starts off by separating our list of monomials into separate levels, so each expression is indexed by the order of the pole at ∞ of that expression. The aim is then to find linear combinations of expressions in any particular level that sit in a lower level. Our list of monomials separates into levels as indicated by the table in Figure 4.1.

Having separated these expressions, the procedure now iterates through the list, from the highest level (in this case, 8) to the lowest level seeking linear

Level	Expressions
8	$f_7a_1^4$,$a_1b_0^2$
6	$f_6a_1^3$, $f_7a_1^2a_2$, b_0b_1
4	$f_5a_1^2$, $f_6a_1a_2$, $f_7a_2^2$, $f_7a_1a_3$
2	f_4a_1, f_5a_2, f_6a_3
0	f_3

Figure 4.1: Monomials indexed by pole-order at ∞

combinations of the expressions from that level that belong in a lower level. When these combinations are found, the resulting expression is added to that (possibly new) lower level. Note that, as any newly created level is always lower than the level currently being processed, and that below 0 only the level $-\infty$ exists[1], this algorithm always terminates.

Figure 4.2 indicates the progress of the distillation as each level is processed. Each entry in the first 'Level' column represents the processing of a particular level, so that anything that appears in the same row was as a result of processing that particular level. The 'Terms' column represents the total number of terms in all expressions under consideration expanded into the corresponding expression in x_i, y_i; e.g., a_1 expands into $x_1 + x_2 + x_3$ in genus 3, and so has 3 terms, but a_3 expands into $x_1x_2x_3$ and has only 1 term. The 'Matrix' column represents the dimension of the matrix used to calculate the basis for the linear combinations of expressions of a particular level that would sit in a lower level. Finally, The 'Added Expressions' columns represent the results of processing the current level; an entry in the 'Expressions' column represents an expression that was obtained from the current level that sits in a lower level, which is indicated in the adjacent 'Level' column.

So, for instance, whilst processing level 8, the expanded expressions under consideration had 205 terms in total; a 3×2 matrix was used to calculate that $a_1b_0^2 - f_7a_1^4$ should sit in level 6. Note in particular that, before level 4 was processed, level 3 did not exist. After level 4 was processed, the next-heighest level was the newly-created level 3, and so that was the next level to be processed.

[1]The level $-\infty$ represents those expressions which are identically 0.

			Added Expressions	
Level	Terms	Matrix	Level	Expressions
8	205	3×2	6	$a_1 b_0^2 - f_7 a_1^4$
6	353	7×4	4	$a_1 b_0^2 - f_7 a_1^4 + f_7 a_1^2 a_2 - f_6 a_1^3$
			4	$b_0 b_1 + f_7 a_1^2 a_2$
4	363	12×6	3	$-f_5 a_1^2 - f_7 a_2^2 + f_7 a_1 a_3 - f_6 a_1^3$ $+f_7 a_1^2 a_2 - f_7 a_1^4 + a_1 b_0^2$
			3	$f_6 a_1 a_2 - f_7 a_1 a_3 + f_7 a_1^2 a_2 + b_0 b_1$
3	264	4×2	2	$-f_5 a_1^2 - f_7 a_2^2 - f_7 a_1 a_3 - f_6 a_1^3$ $+3 f_7 a_1^2 a_2 - f_7 a_1^4 + a_1 b_0^2$ $+2 f_6 a_1 a_2 + 2 b_0 b_1$
2	201	14×4	—	
0	19	3×1	—	

Figure 4.2: Distillation to obtain c_1 in genus 3.

This process continues until finally we reach the state indicated by Figure 4.3; at this point, we can see that c_1 is the new expression in level 2, and our example is complete.

It may be instructive to give some indication of how the size of objects that occur using this algorithm increases as the task becomes more complex. Contrast the output from Figure 4.2 with that from Figure 4.4 to see how, in calculating c_3 (which has the same y-weight and a w-weight greater by only 2), there is a large increase in the size of objects needed. Note that, as the expressions are very long, the 'Expressions' column has been replaced in Figure 4.4 by a column indicating the *number* of expressions that are found to be in a lower level.

Level	Expressions
8	$f_7a_1^4$,$a_1b_0^2$
6	$f_6a_1^3,\ f_7a_1^2a_2,\ b_0b_1,\ -f_7a_1^4+a_1b_0^2$
4	$f_5a_1^2,\ f_6a_1a_2,\ f_7a_2^2,\ f_7a_1a_3,$ $-f_6a_1^3+f_7a_1^2a_2-f_7a_1^4+a_1b_0^2,$ $f_7a_1^2a_2+b_0b_1$
3	$-f_5a_1^2-f_7a_2^2+f_7a_1a_3-f_6a_1^3+f_7a_1^2a_2-f_7a_1^4+a_1b_0^2,$ $f_6a_1a_2-f_7a_1a_3+f_7a_1^2a_2+b_0b_1$
2	$-f_5a_1^2-f_7a_2^2-f_7a_1a_3-f_6a_1^3+3f_7a_1^2a_2$ $\quad -f_7a_1^4+a_1b_0^2+2f_6a_1a_2+2b_0b_1,$ $f_4a_1,\ f_5a_2,\ f_6a_3$
0	f_3

Figure 4.3: Post-distillation: expressions indexed by pole-order at ∞

			Added Expressions	
Level	Terms	Matrix	Level	#(Expressions)
12	828	4×4	10	3
10	1682	7×8	8	6
8	2411	12×14	7	4
			6	6
7	1042	4×4	6	3
6	2178	18×16	5	8
			4	2
5	1366	6×8	4	4
			$-\infty$	2
4	1140	21×12	3	4
3	481	4×4	2	3
2	397	19×6	0	1
			$-\infty$	1
0	38	3×2	$-\infty$	1

Figure 4.4: Distillation to obtain c_3 in genus 3.

Chapter 5

Algorithms for Determining Defining Equations

5.1 Outline of General Strategy

To find the various relations, we simply seek to find linear combinations of products of the a_i, c_i in such a way as to clear the coefficient of $y_i y_j$; then whatever remains may be expressed in terms of the elementary symmetric polynomials in the x_i which, happily, are easily expressed in terms of the a_i.

We first give a systematic way of deriving the quartic defining equation of the Kummer surface in genus 2. We can first look at the set of quartic monomials in $k_0, \ldots, k_3$, as defined in (2.8), and draw up a chart of coefficients of $y_1 y_2$. We find linear dependency, that

$$\begin{aligned} &\kappa_1^2\kappa_3^2 - 4\kappa_0\kappa_2\kappa_3^2 - 4f_0\kappa_0^3\kappa_3 - 2f_1\kappa_0^2\kappa_1\kappa_3 - 4f_2\kappa_0^2\kappa_2\kappa_3 \\ &-2f_3\kappa_0\kappa_1\kappa_2\kappa_3 - 4f_4\kappa_0\kappa_2^2\kappa_3 - 2f_5\kappa_1\kappa_2^2\kappa_3 - 4f_6\kappa_2^3\kappa_3 \end{aligned} \tag{5.1}$$

has coefficient of $y_1 y_2$ equal to 0. The resulting rational function has denomiator a power of δ and numerator entirely in terms of x_1, x_2. Since there is a pole only at $(x, u) = (x_1, -y_1)$ but not at $(x, u) = (x_1, y_1)$, it follows that the denominator must cancel, and we are left with a symmetric polynomial in x_1, x_2, which can be expressed in terms of the elementary symmetric polynomials

$\kappa_0, \kappa_1, \kappa_2 = a_0, a_1, a_2$. This then gives the quartic defining equation (2.13) for the Kummer surface. Note that the defining equation is only quadratic in κ_3, the unique function with denominator δ.

In genus 3, in order to clear the $y_i y_j$ coefficients, we first note that each $y_i y_j$ coefficient is the same in γ_i as the corresponding coefficient in $c_i = \text{wred } \gamma_i$ so it is just as easy (and may be preferable) to perform the initial stage using the γ_i rather than the c_i. Define, as usual, $\delta = (x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$ then the following collections should be sufficient to determine all the required relations.

$$T_1 = \{\delta^4 a_i a_j a_k c_\ell : 3 \geq i \geq j \geq k \geq 0; 3 \geq \ell \geq 0\} \tag{5.2}$$

$$T_2 = \{\delta^4 a_i a_j c_k c_\ell : 3 \geq i \geq j \geq 0; 3 \geq k \geq \ell \geq 0\} \tag{5.3}$$

$$\tag{5.4}$$

Our general strategy is then to run through a chart of coefficients of $y_1 y_2$ and to find linear dependencies between them. Note that, by symmetry, if the coefficient of $y_1 y_2$ is forced to be 0, then so will be the coefficients of $y_2 y_3$ and $y_3 y_1$, giving a numerator entirely in terms of x_1, x_2, x_3. The resulting symmetric polynomial in x_1, x_2, x_3 is then expressed in terms of the elementary symmetric functions a_0, a_1, a_2.

5.2 Defining Equations for $L(2\Theta)$ in Genus 3.

In genus 3, the expressions are too large, and the available monomials are too numerous, to search for linear relationships between the entire set of (for example) quadratic, cubic and quartic monomials. It is helpful to have a technical lemma, analogous to Lemma 3.12.

Lemma 5.1 *Let wt be a weight for which $\mathcal{C} : y^2 = f_n x^n + \ldots + f_0$, δ and each member $d_1, \ldots d_k$ of a basis for $L(n\Theta)$ are homogeneous (where the denominator of each d_i is a power of δ). Let $\phi(d_1, \ldots, d_k) = 0$ be a polynomial relation* *satisfied by $d_1, \ldots, d_k$. Let $\phi_{i_1}, \ldots, \phi_{i_m}$ be the homogeneous pieces of ϕ satisfying $\phi = \phi_{i_1} + \ldots + \phi_{i_m}$ and $wt(\phi_{i_1}) = i_1, \ldots, wt(\phi_{i_m}) = i_m$, where $i_1, \ldots, i_m \in \mathbb{Z}$ are distinct. Then each of $\phi_{i_1}, \ldots, \phi_{i_m}$ is also a relation satisfied by $d_1, \ldots d_k$.*

Proof The same argument applies as in the proof of Lemma 3.12, since polynomial relations satisfied by $d_1, \ldots d_k$ are preserved by linear combinations. $\square$

This means that, when searching for a polynomial relation of degree ℓ, it is not necessary to search for linear dependencies on the set of all monomials of degree ℓ in $d_1, \ldots d_k$; rather, one can slice up the set of such monomials in portions of equal weight, for any weight satisfying the conditions of Lemma 5.1, such that the x-weight and y-weight of Figure 3.1.

After considerable computations, we can exhaustively find every possible quadratic, cubic and quartic relation on the Kummer, and so list a set of defining equations for the $L(2\Theta)$ embedding of the Kummer variety in genus 3, subject to Conjecture 2.4. Redundant equations can be weeded out using the Groebner basis package in Maple. If r_k lies in the variety defined by $r_1, \ldots, r_{k-1}$ then the Groebner basis of $r_1, \ldots, r_{k-1}$ is the same as that of $r_1, \ldots, r_{k-1}, r_k$, and so we can sometimes notice and remove any redundant relations, which we have done from an original list of 41. However, the equations are too large to perform a Groebner basis operation on the whole set of polynomials; we conjecture that there are no dependencies based on interactive sessions in Maple applied to many specialisations of $f_0, \ldots, f_7$. The final result gives the genus 3 equivalent of the single quartic equation (2.13) whch defines the Kummer surface in genus 2.

Theorem 5.2 *Let $a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3$ be the basis for $L(2\Theta)$ in genus 3 given in Theorem 3.13. Then the following* $\mathbf{eq}[\mathbf{i}] = 0$ *generate all relations that are at most quartic in a_0, a_1, a_2, a_3, c_0, c_1, c_2, c_3, and are at most quadratic in c_0, c_1, c_2, c_3.*

$\mathbf{eq}[\mathbf{1}] = a_0c_3 - a_1c_2 - a_2c_1 - a_3c_0 - 2 f_5a_1a_3 + f_5a_2^2 + 2 f_6a_2a_3 + 3 f_7a_3^2.$

$\mathbf{eq}[\mathbf{2}] = -4 a_1a_3c_3^2 + a_2^2c_3^2 + 2 a_2a_3c_2c_3 + 4 a_3^2c_1c_3 + a_3^2c_2^2 - 4 f_0a_0^2a_3c_3 + 4 f_0a_0a_1a_2c_3 + 4 f_0a_0a_1a_3c_2$
$-4 f_0a_0a_2^2c_2 - 4 f_0a_1^3c_3 + 4 f_0a_1^2a_2c_2 + 4 f_0a_1^2a_3c_1 + 8 f_0a_1a_2a_3c_0 - 4 f_0a_2^3c_0 + 2 f_1a_0a_2^2c_3 - 2 f_1a_0a_2a_3c_2$
$-2 f_1a_1^2a_2c_3 + 2 f_1a_1^2a_3c_2 + 4 f_1a_1a_3^2c_0 - 4 f_1a_2^2a_3c_0 - 2 f_2a_0a_3^2c_2 + 4 f_2a_1^2a_3c_3 - 2 f_2a_1a_2^2c_3$

$-4\,f_2a_1{a_3}^2c_1 \;+2\,f_2{a_2}^2a_3c_1 \;-2\,f_2a_2{a_3}^2c_0 \;-4\,f_3a_0{a_3}^2c_3 \;+6\,f_3a_1a_2a_3c_3 \;+2\,f_3a_1{a_3}^2c_2 \;-2\,f_3{a_2}^3c_3$
$-2\,f_3{a_2}^2a_3c_2 \;+4\,f_5{a_3}^3c_2 \;-4\,f_6{a_3}^3c_3 + 8\,f_0f_5a_0a_1{a_3}^2 \;-8\,f_0f_5a_0{a_2}^2a_3 \;+4\,f_0f_5{a_1}^2a_2a_3 \;-4\,f_0f_6{a_1}^2{a_3}^2$
$-8\,f_0f_6a_1{a_2}^2a_3 \;+4\,f_0f_6{a_2}^4 \;-24\,f_0f_7a_1a_2{a_3}^2 \;+12\,f_0f_7{a_2}^3a_3 \;+{f_1}^2{a_0}^2{a_2}^2 \;-2\,{f_1}^2a_0{a_1}^2a_2 \;+{f_1}^2{a_1}^4$
$+2\,f_1f_2{a_0}^2a_2a_3 \;-2\,f_1f_2a_0{a_1}^2a_3 \;-2\,f_1f_2a_0a_1{a_2}^2 \;+2\,f_1f_2{a_1}^3a_2 \;+2\,f_1f_3a_0a_1a_2a_3 \;-2\,f_1f_3a_0{a_2}^3 \;-2\,f_1f_3{a_1}^3a_3$
$+2\,f_1f_3{a_1}^2{a_2}^2 \;-4\,f_1f_5a_0a_2{a_3}^2 \;+4\,f_1f_5{a_1}^2{a_3}^2 \;-4\,f_1f_6a_1a_2{a_3}^2 \;+4\,f_1f_6{a_2}^3a_3 \;-12\,f_1f_7a_1{a_3}^3 \;+12\,f_1f_7{a_2}^2{a_3}^2$
$+{f_2}^2{a_0}^2{a_3}^2 \;-2\,{f_2}^2a_0a_1a_2a_3 \;+{f_2}^2{a_1}^2{a_2}^2 \;+2\,f_2f_3a_0a_1{a_3}^2 \;-2\,f_2f_3a_0{a_2}^2a_3 \;-2\,f_2f_3{a_1}^2a_2a_3 \;+2\,f_2f_3a_1{a_2}^3$
$-4\,f_2f_5a_0{a_3}^3 \;+4\,f_2f_5a_1a_2{a_3}^2 \;-2\,f_2f_5{a_2}^3a_3 \;+4\,f_2f_6a_1{a_3}^3 \;+6\,f_2f_7a_2{a_3}^3 \;+{f_3}^2{a_1}^2{a_3}^2 \;-2\,{f_3}^2a_1{a_2}^2a_3 \;+{f_3}^2{a_2}^4$
$+4\,f_3f_5a_1{a_3}^3 \;-4\,f_3f_5{a_2}^2{a_3}^2 \;+4\,{f_5}^2{a_3}^4.$

$\mathbf{eq[3]} = -\mathbf{a_0a_3{c_3}^2} \;-3\,a_1a_3c_2c_3 \;+{a_2}^2c_2c_3 \;+a_2a_3{c_2}^2 \;-{a_3}^2c_0c_3 \;+{a_3}^2c_1c_2 + 4\,f_0{a_0}^2a_2c_3 \;-4\,f_0a_0{a_1}^2c_3$
$+2\,f_0a_0a_1a_3c_1 \;-2\,f_0a_0{a_2}^2c_1 \;+2\,f_0{a_1}^2a_3c_0 \;-2\,f_0a_1{a_2}^2c_0 \;+f_1{a_0}^2a_3c_3 \;-f_1a_0a_1a_2c_3 \;+f_1a_0{a_2}^2c_2 \;+f_1a_0{a_3}^2c_0$
$-f_1{a_1}^2a_2c_2 \;-f_1{a_1}^2a_3c_1 \;-2\,f_1a_1a_2a_3c_0 \;-2\,f_2a_0{a_2}^2c_3 \;-f_2a_0{a_3}^2c_1 \;+4\,f_2{a_1}^2a_3c_2 \;+3\,f_2a_1a_2a_3c_1 \;+2\,f_2a_1{a_3}^2c_0$
$+f_2{a_2}^3c_1 \;+f_2{a_2}^2a_3c_0 \;+3\,f_3a_0a_2a_3c_3 \;+f_3a_1{a_3}^2c_1 \;-f_3{a_2}^3c_2 \;-2\,f_3{a_2}^2a_3c_1 \;-f_3a_2{a_3}^2c_0 \;+2\,f_4a_0{a_3}^2c_3$
$-3\,f_4a_1a_2a_3c_3 \;-f_4a_1{a_3}^2c_2 \;+f_4{a_2}^3c_3 + f_4{a_2}^2a_3c_2 \;-8\,f_5a_1{a_3}^2c_3 \;+3\,f_5{a_2}^2a_3c_3 \;+4\,f_5a_2{a_3}^2c_2 \;+2\,f_5{a_3}^3c_1$
$+2\,f_6a_2{a_3}^2c_3 \;+5\,f_7{a_3}^3c_3 - 2\,f_0f_5{a_1}^3a_3 \;+2\,f_0f_5{a_1}^2{a_2}^2 \;-4\,f_0f_6{a_1}^2a_2a_3 \;+4\,f_0f_6a_1{a_2}^3 \;-4\,f_0f_7{a_1}^2{a_3}^2$
$+2\,f_0f_7a_1{a_2}^2a_3 \;+2\,f_0f_7{a_2}^4 \;-{f_1}^2{a_0}^2a_1a_2 \;+{f_1}^2a_0{a_1}^3 \;-f_1f_2{a_0}^2a_1a_3 \;-f_1f_2{a_0}^2{a_2}^2 \;+2\,f_1f_2a_0{a_1}^2a_2$
$-f_1f_3a_0{a_1}^2a_3 \;+f_1f_3a_0a_1{a_2}^2 \;-f_1f_4a_0a_1a_2a_3 \;+f_1f_4a_0{a_2}^3 \;+f_1f_4{a_1}^3a_3 \;-f_1f_4{a_1}^2{a_2}^2 \;+f_1f_5a_0{a_2}^2a_3$
$-2\,f_1f_6a_0a_2{a_3}^2 \;+4\,f_1f_6a_1{a_2}^2a_3 \;-3\,f_1f_7a_0{a_3}^3 \;+4\,f_1f_7a_1a_2{a_3}^2 \;+2\,f_1f_7{a_2}^3a_3 \;-{f_2}^2{a_0}^2a_2a_3 \;+{f_2}^2a_0a_1{a_2}^2$
$-f_2f_3a_0a_1a_2a_3 \;+f_2f_3a_0{a_2}^3 \;-f_2f_4a_0a_1{a_3}^2 \;+f_2f_4a_0{a_2}^2a_3 \;+f_2f_4{a_1}^2a_2a_3 \;-f_2f_4a_1{a_2}^3 \;+10\,f_2f_5{a_1}^2{a_3}^2$
$-4\,f_2f_5a_1{a_2}^2a_3 \;-f_2f_5{a_2}^4 \;-4\,f_2f_6a_1a_2{a_3}^2 \;-2\,f_2f_6{a_2}^3a_3 \;-8\,f_2f_7a_1{a_3}^3 \;-f_2f_7{a_2}^2{a_3}^2 \;-f_3f_4{a_1}^2{a_3}^2$
$+2\,f_3f_4a_1{a_2}^2a_3 \;-f_3f_4{a_2}^4 \;-f_3f_5{a_2}^3a_3 \;+2\,f_3f_6{a_2}^2{a_3}^2 \;+3\,f_3f_7a_2{a_3}^3 \;-2\,f_4f_5a_1{a_3}^3 \;+2\,f_4f_5{a_2}^2{a_3}^2$
$+4\,{f_5}^2a_2{a_3}^3.$

$\mathbf{eq[4]} = -4\,a_0a_3c_2c_3 \;+{a_2}^2{c_2}^2 \;+2\,a_2a_3c_1c_2 \;+{a_3}^2{c_1}^2 - 4\,f_0{a_0}^2a_1c_3 \;+4\,f_0a_0a_1a_3c_0 \;-4\,f_0a_0{a_2}^2c_0 \;-4\,f_1{a_0}^2a_2c_3$
$+2\,f_1a_0a_1a_2c_2 \;-2\,f_1a_0a_1a_3c_1 \;+4\,f_1a_0{a_2}^2c_1 \;+2\,f_2{a_0}^2a_3c_3 \;+2\,f_2a_0a_1a_3c_2 \;-2\,f_2a_0{a_2}^2c_2 \;-2\,f_2a_0{a_3}^2c_0$
$-6\,f_4a_0a_2a_3c_3 \;-2\,f_4a_1{a_3}^2c_1 \;+2\,f_4{a_2}^3c_2 \;+4\,f_4{a_2}^2a_3c_1 \;+2\,f_4a_2{a_3}^2c_0 \;-12\,f_5a_0{a_3}^2c_3 \;+4\,f_5{a_2}^2a_3c_2$
$+4\,f_5a_2{a_3}^2c_1 - 4\,f_0f_5a_0{a_1}^2a_3 \;+4\,f_0f_5a_0a_1{a_2}^2 \;-8\,f_0f_6a_0a_1a_2a_3 \;+8\,f_0f_6a_0{a_2}^3 - 12\,f_0f_7a_0a_1{a_3}^2$
$+12\,f_0f_7a_0{a_2}^2a_3 \;+{f_1}^2{a_0}^2{a_1}^2 \;+2\,f_1f_2{a_0}^2a_1a_2 \;+2\,f_1f_4a_0{a_1}^2a_3 \;-2\,f_1f_4a_0a_1{a_2}^2 \;+8\,f_1f_5a_0a_1a_2a_3$
$-4\,f_1f_5a_0{a_2}^3 \;+{f_2}^2{a_0}^2{a_2}^2 \;+2\,f_2f_4a_0a_1a_2a_3 \;-2\,f_2f_4a_0{a_2}^3 \;+8\,f_2f_5a_0a_1{a_3}^2 \;-6\,f_2f_5a_0{a_2}^2a_3 \;+4\,f_2f_6a_0a_2{a_3}^2$
$+6\,f_2f_7a_0{a_3}^3 \;+{f_4}^2{a_1}^2{a_3}^2 \;-2\,{f_4}^2a_1{a_2}^2a_3 \;+{f_4}^2{a_2}^4 \;+2\,f_4f_5{a_2}^3a_3 \;-4\,f_4f_6{a_2}^2{a_3}^2 \;-6\,f_4f_7a_2{a_3}^3 \;+4\,{f_5}^2{a_2}^2{a_3}^2.$

$\mathbf{eq[5]} = -2\,a_0a_2{c_3}^2 \;-8\,a_0a_3c_2c_3 \;+2\,a_1a_2c_2c_3 \;+4\,a_1a_3c_1c_3 \;+{a_2}^2{c_2}^2 \;+2\,{a_3}^2c_0c_2 \;-3\,{a_3}^2{c_1}^2 - 8\,f_0{a_0}^2a_1c_3$
$+8\,f_0{a_0}^2a_2c_2 \;+4\,f_0{a_0}^2a_3c_1 \;-4\,f_0a_0{a_1}^2c_2 \;+4\,f_0a_0a_1a_3c_0 \;+4\,f_0a_0{a_2}^2c_0 \;-4\,f_0{a_1}^2a_2c_0 \;+2\,f_1{a_0}^2a_3c_2$
$-4\,f_1a_0a_1a_2c_2 \;-2\,f_1a_0a_1a_3c_1 \;-4\,f_1a_0{a_2}^2c_1 \;+2\,f_1{a_1}^2a_2c_1 \;-2\,f_1{a_1}^2a_3c_0 \;+2\,f_2a_0a_1a_2c_3 \;+8\,f_2a_0a_1a_3c_2$
$-2\,f_2a_0{a_2}^2c_2 \;+2\,f_2a_0{a_3}^2c_0 \;-2\,f_2{a_1}^2a_2c_2 \;+2\,f_3a_0{a_2}^2c_3 \;+6\,f_3a_0a_2a_3c_2 \;+4\,f_3a_0{a_3}^2c_1 \;-4\,f_3{a_1}^2a_3c_2$
$-6\,f_3a_1a_2a_3c_1 \;-2\,f_3a_1{a_3}^2c_0 \;+2\,f_3{a_2}^3c_1 \;+2\,f_3{a_2}^2a_3c_0 \;-6\,f_4a_0a_2a_3c_3 \;+2\,f_4a_0{a_3}^2c_2 \;-4\,f_4{a_1}^2a_3c_3$
$+2\,f_4a_1{a_2}^2c_3 \;+2\,f_4a_1{a_3}^2c_1 \;+2\,f_4{a_2}^3c_2 \;+2\,f_4{a_2}^2a_3c_1 \;+4\,f_4a_2{a_3}^2c_0 \;-16\,f_5a_0{a_3}^2c_3 \;+2\,f_5a_1a_2a_3c_3$
$-6\,f_5a_1{a_3}^2c_2 \;+6\,f_5{a_2}^2a_3c_2 \;+4\,f_6{a_3}^3c_1 \;+2\,f_7a_2{a_3}^2c_3 \;-4\,f_7{a_3}^3c_2 + 16\,f_0f_5{a_0}^2a_2a_3 \;-16\,f_0f_5a_0{a_1}^2a_3$
$+4\,f_0f_5{a_1}^3a_2 \;-16\,f_0f_6a_0a_1a_2a_3 \;+8\,f_0f_6{a_1}^2{a_2}^2 \;-8\,f_0f_7a_0a_1{a_3}^2 \;-16\,f_0f_7a_0{a_2}^2a_3 \;+8\,f_0f_7{a_1}^2a_2a_3$
$+4\,f_0f_7a_1{a_2}^3 \;-2\,{f_1}^2{a_0}^3a_2 \;+3\,{f_1}^2{a_0}^2{a_1}^2 \;-2\,f_1f_2{a_0}^3a_3 \;+4\,f_1f_2{a_0}^2a_1a_2 \;-2\,f_1f_3{a_0}^2a_1a_3 \;+4\,f_1f_3{a_0}^2{a_2}^2$
$-2\,f_1f_3a_0{a_1}^2a_2 \;-2\,f_1f_4{a_0}^2a_2a_3 \;+4\,f_1f_4a_0{a_1}^2a_3 \;-2\,f_1f_4{a_1}^3a_2 \;+4\,f_1f_5{a_0}^2{a_3}^2 \;-6\,f_1f_5a_0a_1a_2a_3 \;+4\,f_1f_5a_0{a_2}^3$
$+2\,f_1f_5{a_1}^3a_3 \;-2\,f_1f_5{a_1}^2{a_2}^2 \;-4\,f_1f_6a_0a_1{a_3}^2 \;+8\,f_1f_6a_0{a_2}^2a_3 \;+4\,f_1f_6{a_1}^2a_2a_3 \;-2\,f_1f_7a_0a_2{a_3}^2 \;+4\,f_1f_7{a_1}^2{a_3}^2$
$+4\,f_1f_7a_1{a_2}^2a_3 \;+{f_2}^2{a_0}^2{a_2}^2 \;+2\,f_2f_3{a_0}^2a_2a_3 \;-2\,f_2f_3a_0a_1{a_2}^2 \;-2\,f_2f_4{a_0}^2{a_3}^2 \;+6\,f_2f_4a_0a_1a_2a_3 \;-2\,f_2f_4a_0{a_2}^3$
$-2\,f_2f_4{a_1}^2{a_2}^2 \;+22\,f_2f_5a_0a_1{a_3}^2 \;-6\,f_2f_5a_0{a_2}^2a_3 \;-6\,f_2f_5{a_1}^2a_2a_3 \;-8\,f_2f_7a_0{a_3}^3 \;+2\,f_2f_7a_1a_2{a_3}^2$
$+2\,{f_3}^2a_0a_1a_2a_3 \;-2\,{f_3}^2a_0{a_2}^3 \;-2\,f_3f_4a_0a_1{a_3}^2 \;+2\,f_3f_4a_0{a_2}^2a_3 \;+2\,f_3f_4{a_1}^2a_2a_3 \;-2\,f_3f_4a_1{a_2}^3$
$+12\,f_3f_5a_0a_2{a_3}^2 \;-10\,f_3f_5{a_1}^2{a_3}^2 \;+4\,f_3f_5a_1{a_2}^2a_3 \;-2\,f_3f_5{a_2}^4 \;+4\,f_3f_6a_1a_2{a_3}^2 \;-4\,f_3f_6{a_2}^3a_3 \;+8\,f_3f_7a_1{a_3}^3$
$-8\,f_3f_7{a_2}^2{a_3}^2 \;+{f_4}^2{a_1}^2{a_3}^2 \;-2\,{f_4}^2a_1{a_2}^2a_3 \;+{f_4}^2{a_2}^4 \;+4\,f_4f_5a_0{a_3}^3 \;-4\,f_4f_5a_1a_2{a_3}^2 \;+4\,f_4f_5{a_2}^3a_3$
$-4\,f_4f_6a_1{a_3}^3 \;-4\,f_4f_6{a_2}^2{a_3}^2 \;-12\,f_4f_7a_2{a_3}^3 \;-12\,{f_5}^2a_1{a_3}^3 \;+12\,{f_5}^2{a_2}^2{a_3}^2 \;+8\,f_5f_6a_2{a_3}^3 \;+4\,f_5f_7{a_3}^4.$

$\mathbf{eq[6]} = 2\,a_0a_3c_2c_3 \;+2\,a_1a_3{c_2}^2 \;-{a_2}^2{c_2}^2 \;+2\,{a_3}^2c_0c_2 \;-{a_3}^2{c_1}^2 + 4\,f_0{a_0}^2a_1c_3 \;-4\,f_0a_0a_1a_3c_0 \;+4\,f_0a_0{a_2}^2c_0$
$+4\,f_1{a_0}^2a_2c_3 \;-2\,f_1a_0a_1a_2c_2 \;+2\,f_1a_0a_1a_3c_1 \;-4\,f_1a_0{a_2}^2c_1 \;-2\,f_2{a_0}^2a_3c_3 \;-2\,f_2a_0a_1a_3c_2 \;+2\,f_2a_0{a_2}^2c_2$
$+2\,f_2a_0{a_3}^2c_0 \;+6\,f_4a_0a_2a_3c_3 \;+2\,f_4a_1{a_3}^2c_1 \;-2\,f_4{a_2}^3c_2 \;-4\,f_4{a_2}^2a_3c_1 \;-2\,f_4a_2{a_3}^2c_0 \;+12\,f_5a_0{a_3}^2c_3$

$+4\,f_5a_1{a_3}^2c_2\ -6\,f_5{a_2}^2a_3c_2\ -4\,f_5a_2{a_3}^2c_1\ -4\,f_6a_2{a_3}^2c_2\ -6\,f_7{a_3}^3c_2+4\,f_0f_5a_0{a_1}^2a_3\ -4\,f_0f_5a_0a_1{a_2}^2$
$+8\,f_0f_6a_0a_1a_2a_3\ -8\,f_0f_6a_0{a_2}^3\ +12\,f_0f_7a_0a_1{a_3}^2\ -12\,f_0f_7a_0{a_2}^2a_3\ -{f_1}^2{a_0}^2{a_1}^2\ -2\,f_1f_2{a_0}^2a_1a_2$
$-2\,f_1f_4a_0{a_1}^2a_3\ +2\,f_1f_4a_0a_1{a_2}^2\ -8\,f_1f_5a_0a_1a_2a_3\ +4\,f_1f_5a_0{a_2}^3\ -{f_2}^2{a_0}^2{a_2}^2\ -2\,f_2f_4a_0a_1a_2a_3\ +2\,f_2f_4a_0{a_2}^3$
$-8\,f_2f_5a_0a_1{a_3}^2\ +6\,f_2f_5a_0{a_2}^2a_3\ -4\,f_2f_6a_0a_2{a_3}^2\ -6\,f_2f_7a_0{a_3}^3\ -{f_4}^2{a_1}^2{a_3}^2\ +2\,{f_4}^2a_1{a_2}^2a_3\ -{f_4}^2{a_2}^4$
$-2\,f_4f_5{a_2}^3a_3\ +4\,f_4f_6{a_2}^2{a_3}^2\ +6\,f_4f_7a_2{a_3}^3\ -4\,{f_5}^2{a_2}^2{a_3}^2.$

$\mathbf{eq[7]}=8\,a_0a_3c_2c_3\ -4\,a_1a_3c_1c_3\ +2\,{a_2}^2c_1c_3\ -{a_2}^2{c_2}^2\ +2\,a_2a_3c_0c_3\ -2\,{a_3}^2c_0c_2\ +3\,{a_3}^2{c_1}^2+8\,f_0{a_0}^2a_1c_3$
$-8\,f_0{a_0}^2a_2c_2\ -4\,f_0{a_0}^2a_3c_1\ +4\,f_0a_0{a_1}^2c_2\ -4\,f_0a_0a_1a_3c_0\ -4\,f_0a_0{a_2}^2c_0\ +4\,f_0{a_1}^2a_2c_0\ -2\,f_1{a_0}^2a_3c_2$
$+4\,f_1a_0a_1a_2c_2\ +2\,f_1a_0a_1a_3c_1\ +4\,f_1a_0{a_2}^2c_1\ -2\,f_1{a_1}^2a_2c_1\ +2\,f_1{a_1}^2a_3c_0\ -2\,f_2a_0a_1a_2c_3\ -8\,f_2a_0a_1a_3c_2$
$+2\,f_2a_0{a_2}^2c_2\ -2\,f_2a_0{a_3}^2c_0\ +2\,f_2{a_1}^2a_2c_2\ -2\,f_3a_0{a_2}^2c_3\ -6\,f_3a_0a_2a_3c_2\ -4\,f_3a_0{a_3}^2c_1\ +4\,f_3{a_1}^2a_3c_2$
$+6\,f_3a_1a_2a_3c_1\ +2\,f_3a_1{a_3}^2c_0\ -2\,f_3{a_2}^3c_1\ -2\,f_3{a_2}^2a_3c_0\ +6\,f_4a_0a_2a_3c_3\ -2\,f_4a_0{a_3}^2c_2\ +4\,f_4{a_1}^2a_3c_3$
$-2\,f_4a_1{a_2}^2c_3\ -2\,f_4a_1{a_3}^2c_1\ -2\,f_4{a_2}^3c_2\ -2\,f_4{a_2}^2a_3c_1\ -4\,f_4a_2{a_3}^2c_0\ +16\,f_5a_0{a_3}^2c_3\ +2\,f_5a_1a_2a_3c_3$
$+6\,f_5a_1{a_3}^2c_2\ -2\,f_5{a_2}^3c_3\ -6\,f_5{a_2}^2a_3c_2\ -4\,f_6{a_2}^2a_3c_3\ -4\,f_6{a_3}^3c_1\ -8\,f_7a_2{a_3}^2c_3\ +4\,f_7{a_3}^3c_2-16\,f_0f_5{a_0}^2a_2a_3$
$+16\,f_0f_5a_0{a_1}^2a_3\ -4\,f_0f_5{a_1}^3a_2\ +16\,f_0f_6a_0a_1a_2a_3\ -8\,f_0f_6{a_1}^2{a_2}^2\ +8\,f_0f_7a_0a_1{a_3}^2\ +16\,f_0f_7a_0{a_2}^2a_3$
$-8\,f_0f_7{a_1}^2a_2a_3\ -4\,f_0f_7a_1{a_2}^3\ +2\,{f_1}^2{a_0}^3a_2\ -3\,{f_1}^2{a_0}^2{a_1}^2\ +2\,f_1f_2{a_0}^3a_3\ -4\,f_1f_2{a_0}^2a_1a_2\ +2\,f_1f_3{a_0}^2a_1a_3$
$-4\,f_1f_3{a_0}^2{a_2}^2\ +2\,f_1f_3a_0{a_1}^2a_2\ +2\,f_1f_4{a_0}^2a_2a_3\ -4\,f_1f_4a_0{a_1}^2a_3\ +2\,f_1f_4{a_1}^3a_2\ -4\,f_1f_5{a_0}^2{a_3}^2$
$+6\,f_1f_5a_0a_1a_2a_3\ -4\,f_1f_5a_0{a_2}^3\ -2\,f_1f_5{a_1}^3a_3\ +2\,f_1f_5{a_1}^2{a_2}^2\ +4\,f_1f_6a_0a_1{a_3}^2\ -8\,f_1f_6a_0{a_2}^2a_3\ -4\,f_1f_6{a_1}^2a_2a_3$
$+2\,f_1f_7a_0a_2{a_3}^2\ -4\,f_1f_7{a_1}^2{a_3}^2\ -4\,f_1f_7a_1{a_2}^2a_3\ -{f_2}^2{a_0}^2{a_2}^2\ -2\,f_2f_3{a_0}^2a_2a_3\ +2\,f_2f_3a_0a_1{a_2}^2\ +2\,f_2f_4{a_0}^2{a_3}^2$
$-6\,f_2f_4a_0a_1a_2a_3\ +2\,f_2f_4a_0{a_2}^3\ +2\,f_2f_4{a_1}^2{a_2}^2\ -22\,f_2f_5a_0a_1{a_3}^2\ +6\,f_2f_5a_0{a_2}^2a_3\ +6\,f_2f_5{a_1}^2a_2a_3$
$+8\,f_2f_7a_0{a_3}^3\ -2\,f_2f_7a_1a_2{a_3}^2\ -2\,{f_3}^2a_0a_1a_2a_3\ +2\,{f_3}^2a_0{a_2}^3\ +2\,f_3f_4a_0a_1{a_3}^2\ -2\,f_3f_4a_0{a_2}^2a_3\ -2\,f_3f_4{a_1}^2a_2a_3$
$+2\,f_3f_4a_1{a_2}^3\ -12\,f_3f_5a_0a_2{a_3}^2\ +10\,f_3f_5{a_1}^2{a_3}^2\ -4\,f_3f_5a_1{a_2}^2a_3\ +2\,f_3f_5{a_2}^4\ -4\,f_3f_6a_1a_2{a_3}^2\ +4\,f_3f_6{a_2}^3a_3$
$-8\,f_3f_7a_1{a_3}^3\ +8\,f_3f_7{a_2}^2{a_3}^2\ -{f_4}^2{a_1}^2{a_3}^2\ +2\,{f_4}^2a_1{a_2}^2a_3\ -{f_4}^2{a_2}^4\ -4\,f_4f_5a_0{a_3}^3\ +4\,f_4f_5a_1a_2{a_3}^2$
$-4\,f_4f_5{a_2}^3a_3\ +4\,f_4f_6a_1{a_3}^3\ +4\,f_4f_6{a_2}^2{a_3}^2\ +12\,f_4f_7a_2{a_3}^3\ +12\,{f_5}^2a_1{a_3}^3\ -12\,{f_5}^2{a_2}^2{a_3}^2\ -8\,f_5f_6a_2{a_3}^3$
$-4\,f_5f_7{a_3}^4.$

$\mathbf{eq[8]}=-4\,{a_0}^2c_0c_2\ +{a_0}^2{c_1}^2\ -2\,a_0a_1c_0c_1\ -4\,a_0a_2{c_0}^2\ +{a_1}^2{c_0}^2-4\,f_5{a_0}^2a_2c_1\ -12\,f_5{a_0}^2a_3c_0\ +2\,f_5a_0{a_1}^2c_1$
$+8\,f_5a_0a_1a_2c_0\ -2\,f_5{a_1}^3c_0\ +8\,f_6{a_0}^2a_2c_2\ -4\,f_6a_0{a_1}^2c_2+12\,f_6a_0{a_2}^2c_0\ -4\,f_6{a_1}^2a_2c_0\ +28\,f_7{a_0}^2a_2c_3$
$+12\,f_7{a_0}^2a_3c_2\ -4\,f_7a_0{a_1}^2c_3\ -24\,f_7a_0a_1a_2c_2\ +8\,f_7a_0a_1a_3c_1\ -26\,f_7a_0{a_2}^2c_1\ -4\,f_7{a_1}^2a_3c_0$
$-2\,f_7a_1{a_2}^2c_0+4\,{f_5}^2{a_0}^2{a_2}^2\ -4\,{f_5}^2a_0{a_1}^2a_2\ +{f_5}^2{a_1}^4\ +24\,f_5f_6{a_0}^2a_2a_3\ -12\,f_5f_6a_0{a_1}^2a_3\ -8\,f_5f_6a_0a_1{a_2}^2$
$+4\,f_5f_6{a_1}^3a_2\ +36\,f_5f_7{a_0}^2{a_3}^2\ -72\,f_5f_7a_0a_1a_2a_3\ +24\,f_5f_7a_0{a_2}^3\ +4\,f_5f_7{a_1}^3a_3\ +2\,f_5f_7{a_1}^2{a_2}^2\ -8\,{f_6}^2a_0{a_2}^3$
$+4\,{f_6}^2{a_1}^2{a_2}^2\ +12\,f_6f_7a_0{a_2}^2a_3\ +8\,f_6f_7{a_1}^2a_2a_3\ +4\,f_6f_7a_1{a_2}^3\ +36\,{f_7}^2a_0a_2{a_3}^2\ +4\,{f_7}^2{a_1}^2{a_3}^2\ +4\,{f_7}^2a_1{a_2}^2a_3$
$+{f_7}^2{a_2}^4.$

$\mathbf{eq[9]}=a_0a_2c_2c_3\ +2\,a_0a_3{c_2}^2\ -a_1a_2{c_2}^2-2\,a_1a_3c_1c_2\ -a_2a_3{c_1}^2\ -{a_3}^2c_0c_1+4\,f_0{a_0}^2a_1c_2\ +2\,f_0{a_0}^2a_2c_1$
$+2\,f_0a_0a_1a_2c_0\ -f_1{a_0}^2a_1c_3\ +f_1{a_0}^2a_2c_2\ +f_1{a_0}^2a_3c_1\ +f_1a_0{a_1}^2c_2\ +2\,f_1a_0a_1a_3c_0\ -f_2{a_0}^2a_2c_3\ -2\,f_2{a_0}^2a_3c_2$
$+f_2a_0a_1a_2c_2\ +f_3{a_0}^2a_3c_3\ +f_3a_0a_1a_3c_2\ -f_3a_0{a_2}^2c_2\ -f_3a_0{a_3}^2c_0\ -f_4a_0{a_2}^2c_3\ +3\,f_4a_0a_2a_3c_2\ +f_4a_0{a_3}^2c_1$
$+2\,f_4{a_1}^2a_3c_2\ +f_4a_1{a_3}^2c_0\ +2\,f_4{a_2}^3c_1\ +2\,f_4{a_2}^2a_3c_0\ -f_5a_0a_2a_3c_3\ +10\,f_5a_0{a_3}^2c_2\ -5\,f_5a_1{a_3}^2c_1\ +6\,f_5{a_2}^2a_3c_1$
$+5\,f_5a_2{a_3}^2c_0\ +2\,f_6a_2{a_3}^2c_1\ +f_7a_2{a_3}^2c_2\ +4\,f_7{a_3}^3c_1+12\,f_0f_5{a_0}^2a_1a_3\ -4\,f_0f_5{a_0}^2{a_2}^2\ -2\,f_0f_5a_0{a_1}^2a_2$
$-4\,f_0f_6a_0a_1{a_2}^2\ -8\,f_0f_7a_0a_1a_2a_3\ +2\,f_0f_7a_0{a_2}^3\ -{f_1}^2{a_0}^3a_1\ -f_1f_2{a_0}^3a_2\ +f_1f_3{a_0}^2a_1a_2\ -2\,f_1f_4{a_0}^2a_1a_3$
$+f_1f_4{a_0}^2{a_2}^2\ +f_1f_4a_0{a_1}^2a_2\ +f_1f_5a_0{a_1}^2a_3\ -4\,f_1f_6a_0a_1a_2a_3\ -7\,f_1f_7a_0a_1{a_3}^2\ +2\,f_1f_7a_0{a_2}^2a_3\ +f_2f_3{a_0}^2{a_2}^2$
$-f_2f_4{a_0}^2a_2a_3\ +f_2f_4a_0a_1{a_2}^2\ -6\,f_2f_5{a_0}^2{a_3}^2\ +3\,f_2f_5a_0a_1a_2a_3\ +f_2f_7a_0a_2{a_3}^2\ +f_3f_4a_0a_1a_2a_3\ -f_3f_4a_0{a_2}^3$
$+4\,f_3f_5a_0a_1{a_3}^2\ -3\,f_3f_5a_0{a_2}^2a_3\ +2\,f_3f_6a_0a_2{a_3}^2\ +3\,f_3f_7a_0{a_3}^3\ -{f_4}^2a_0a_1{a_3}^2\ +{f_4}^2a_0{a_2}^2a_3\ +{f_4}^2{a_1}^2a_2a_3$
$-{f_4}^2a_1{a_2}^3\ +6\,f_4f_5a_0a_2{a_3}^2\ +5\,f_4f_5{a_1}^2{a_3}^2\ -2\,f_4f_5a_1{a_2}^2a_3-2\,f_4f_5{a_2}^4\ -2\,f_4f_6a_1a_2{a_3}^2\ -4\,f_4f_6{a_2}^3a_3$
$-4\,f_4f_7a_1{a_3}^3\ -5\,f_4f_7{a_2}^2{a_3}^2\ +12\,{f_5}^2a_0{a_3}^3\ +4\,{f_5}^2a_1a_2{a_3}^2\ -5\,{f_5}^2{a_2}^3a_3\ -10\,f_5f_6{a_2}^2{a_3}^2\ -13\,f_5f_7a_2{a_3}^3.$

$\mathbf{eq[10]}=-a_0a_1{c_3}^2\ +3\,a_0a_3{c_2}^2\ +{a_1}^2c_2c_3-a_1a_2{c_2}^2\ -a_1a_3c_0c_3\ +a_1a_3c_1c_2\ +2\,a_2a_3c_0c_2\ +a_2a_3{c_1}^2$
$+3\,{a_3}^2c_0c_1+4\,f_0{a_0}^2a_1c_2\ -4\,f_0{a_0}^2a_3c_0\ +2\,f_0a_0{a_1}^2c_1\ +8\,f_0a_0a_1a_2c_0\ -2\,f_0{a_1}^3c_0\ -2\,f_1{a_0}^2a_1c_3\ +f_1{a_0}^2a_3c_1$
$+3\,f_1a_0{a_1}^2c_2\ +5\,f_1a_0a_1a_3c_0\ +f_1{a_1}^3c_1\ -2\,f_2{a_0}^2a_2c_3\ -4\,f_2{a_0}^2a_3c_2\ +4\,f_2a_0a_1a_2c_2\ -3\,f_2a_0a_1a_3c_1$
$+2\,f_2a_0{a_2}^2c_1\ +f_2{a_1}^2a_2c_1\ +2\,f_2{a_1}^2a_3c_0\ -f_3{a_0}^2a_3c_3\ +2\,f_3a_0a_1a_2c_3\ +f_3a_0a_1a_3c_2\ -3\,f_3a_0{a_3}^2c_0\ -f_3{a_1}^2a_2c_2$
$-f_3{a_1}^2a_3c_1\ +f_3a_1a_2a_3c_0\ -2\,f_4a_0{a_2}^2c_3\ +4\,f_4a_0a_2a_3c_2+f_4a_0{a_3}^2c_1\ +f_4{a_1}^2a_2c_3\ +f_4{a_1}^2a_3c_2\ -f_4a_1{a_3}^2c_0$
$+2\,f_4{a_2}^3c_1\ +4\,f_4{a_2}^2a_3c_0\ -8\,f_5a_0a_2a_3c_3\ +14\,f_5a_0{a_3}^2c_2\ +4\,f_5{a_1}^2a_3c_3\ +f_5a_1{a_3}^2c_1\ +5\,f_5{a_2}^2a_3c_1$
$+10\,f_5a_2{a_3}^2c_0\ -4\,f_6a_0{a_3}^2c_3\ +2\,f_6a_1a_2a_3c_3\ -2\,f_6{a_2}^2a_3c_2\ -4\,f_6a_2{a_3}^2c_1\ +f_7a_1{a_3}^2c_3\ +f_7{a_2}^2a_3c_3$

$-4\,f_7a_2{a_3}^2c_2\ -8\,f_7{a_3}^3c_1+12\,f_0f_5{a_0}^2a_1a_3\ -8\,f_0f_5a_0{a_1}^2a_2\ +2\,f_0f_5{a_1}^4\ +8\,f_0f_6{a_0}^2a_2a_3\ -4\,f_0f_6a_0{a_1}^2a_3$
$-12\,f_0f_6a_0a_1{a_2}^2\ +4\,f_0f_6{a_1}^3a_2\ +12\,f_0f_7{a_0}^2{a_3}^2\ -24\,f_0f_7a_0a_1a_2a_3\ +4\,f_0f_7{a_1}^3a_3\ +2\,f_0f_7{a_1}^2{a_2}^2\ -{f_1}^2{a_0}^3a_1$
$-f_1f_2{a_0}^2{a_1}^2\ +2\,f_1f_3{a_0}^2a_1a_2\ -f_1f_3a_0{a_1}^3\ -2\,f_1f_4{a_0}^2a_1a_3\ +3\,f_1f_4a_0{a_1}^2a_2\ -f_1f_4{a_1}^4\ -2\,f_1f_5{a_0}^2a_2a_3$
$+5\,f_1f_5a_0{a_1}^2a_3\ -f_1f_5{a_1}^3a_2\ -8\,f_1f_6a_0a_1a_2a_3\ -16\,f_1f_7a_0a_1{a_3}^2+f_1f_7a_0{a_2}^2a_3\ +f_1f_7{a_1}^2a_2a_3\ +{f_2}^2{a_0}^3a_3$
$-{f_2}^2{a_0}^2a_1a_2\ +2\,f_2f_3{a_0}^2a_1a_3\ -f_2f_3a_0{a_1}^2a_2\ -2\,f_2f_4{a_0}^2a_2a_3\ +f_2f_4a_0{a_1}^2a_3\ +2\,f_2f_4a_0a_1{a_2}^2\ -f_2f_4{a_1}^3a_2$
$-10\,f_2f_5{a_0}^2{a_3}^2\ +14\,f_2f_5a_0a_1a_2a_3\ -2\,f_2f_5a_0{a_2}^3\ -2\,f_2f_5{a_1}^3a_3\ -f_2f_5{a_1}^2{a_2}^2\ +4\,f_2f_6a_0a_1{a_3}^2\ -2\,f_2f_6a_0{a_2}^2a_3$
$-4\,f_2f_6{a_1}^2a_2a_3\ -4\,f_2f_7{a_1}^2{a_3}^2\ -f_2f_7a_1{a_2}^2a_3\ +{f_3}^2a_0{a_1}^2a_3\ -{f_3}^2a_0a_1{a_2}^2\ +f_3f_4{a_1}^3a_3\ -f_3f_4{a_1}^2{a_2}^2$
$+4\,f_3f_5a_0a_1{a_3}^2\ +f_3f_5a_0{a_2}^2a_3\ -3\,f_3f_5{a_1}^2a_2a_3\ +6\,f_3f_6a_0a_2{a_3}^2\ -2\,f_3f_6a_1{a_2}^2a_3\ +9\,f_3f_7a_0{a_3}^3$
$-2\,f_3f_7a_1a_2{a_3}^2\ -f_3f_7{a_2}^3a_3\ -{f_4}^2a_0a_1{a_3}^2\ +{f_4}^2a_0{a_2}^2a_3\ +{f_4}^2{a_1}^2a_2a_3\ -{f_4}^2a_1{a_2}^3\ +8\,f_4f_5a_0a_2{a_3}^2$
$+3\,f_4f_5{a_1}^2{a_3}^2\ -2\,f_4f_5a_1{a_2}^2a_3\ -2\,f_4f_5{a_2}^4\ -6\,f_4f_6{a_2}^3a_3\ +2\,f_4f_7a_1{a_3}^3\ -11\,f_4f_7{a_2}^2{a_3}^2\ +16\,{f_5}^2a_0{a_3}^3$
$+4\,{f_5}^2a_1a_2{a_3}^2\ -6\,{f_5}^2{a_2}^3a_3\ -16\,f_5f_6{a_2}^2{a_3}^2\ -26\,f_5f_7a_2{a_3}^3.$

$\mathbf{eq}[\mathbf{11}]=2\,a_0a_3{c_2}^2\ -2\,a_1a_3c_1c_2\ +{a_2}^2c_1c_2\ +a_2a_3c_0c_2\ -a_2a_3{c_1}^2\ -{a_3}^2c_0c_1+4\,f_0{a_0}^2a_1c_2\ +2\,f_0{a_0}^2a_2c_1$
$+2\,f_0a_0a_1a_2c_0\ -f_1{a_0}^2a_1c_3\ +f_1{a_0}^2a_2c_2\ +f_1{a_0}^2a_3c_1\ +f_1a_0{a_1}^2c_2\ +2\,f_1a_0a_1a_3c_0\ -f_2{a_0}^2a_2c_3\ -2\,f_2{a_0}^2a_3c_2$
$+f_2a_0a_1a_2c_2\ +f_3{a_0}^2a_3c_3\ +f_3a_0a_1a_3c_2\ -f_3a_0{a_2}^2c_2\ -f_3a_0{a_3}^2c_0\ -f_4a_0{a_2}^2c_3\ +3\,f_4a_0a_2a_3c_2\ +f_4a_0{a_3}^2c_1$
$+2\,f_4{a_1}^2a_3c_2\ +f_4a_1{a_3}^2c_0\ +2\,f_4{a_2}^3c_1\ +2\,f_4{a_2}^2a_3c_0\ +f_5a_0a_2a_3c_3\ +10\,f_5a_0{a_3}^2c_2\ -5\,f_5a_1{a_3}^2c_1\ -f_5{a_2}^3c_2$
$+4\,f_5{a_2}^2a_3c_1\ +3\,f_5a_2{a_3}^2c_0\ -2\,f_6{a_2}^2a_3c_2\ +2\,f_6a_2{a_3}^2c_1\ -2\,f_7a_2{a_3}^2c_2\ +4\,f_7{a_3}^3c_1+12\,f_0f_5{a_0}^2a_1a_3$
$-4\,f_0f_5{a_0}^2{a_2}^2\ -2\,f_0f_5a_0{a_1}^2a_2\ -4\,f_0f_6a_0a_1{a_2}^2\ -8\,f_0f_7a_0a_1a_2a_3\ +2\,f_0f_7a_0{a_2}^3\ -{f_1}^2{a_0}^3a_1\ -f_1f_2{a_0}^3a_2$
$+f_1f_3{a_0}^2a_1a_2\ -2\,f_1f_4{a_0}^2a_1a_3\ +f_1f_4{a_0}^2{a_2}^2\ +f_1f_4a_0{a_1}^2a_2\ +f_1f_5a_0{a_1}^2a_3\ -4\,f_1f_6a_0a_1a_2a_3\ -7\,f_1f_7a_0a_1{a_3}^2$
$+2\,f_1f_7a_0{a_2}^2a_3\ +f_2f_3{a_0}^2{a_2}^2\ -f_2f_4{a_0}^2a_2a_3\ +f_2f_4a_0a_1{a_2}^2\ -6\,f_2f_5{a_0}^2{a_3}^2\ +3\,f_2f_5a_0a_1a_2a_3\ +f_2f_7a_0a_2{a_3}^2$
$+f_3f_4a_0a_1a_2a_3\ -f_3f_4a_0{a_2}^3\ +4\,f_3f_5a_0a_1{a_3}^2\ -3\,f_3f_5a_0{a_2}^2a_3\ +2\,f_3f_6a_0a_2{a_3}^2\ +3\,f_3f_7a_0{a_3}^3\ -{f_4}^2a_0a_1{a_3}^2$
$+{f_4}^2a_0{a_2}^2a_3\ +{f_4}^2{a_1}^2a_2a_3\ -{f_4}^2a_1{a_2}^3\ +6\,f_4f_5a_0a_2{a_3}^2\ +5\,f_4f_5{a_1}^2{a_3}^2\ -2\,f_4f_5a_1{a_2}^2a_3-2\,f_4f_5{a_2}^4$
$-2\,f_4f_6a_1a_2{a_3}^2\ -4\,f_4f_6{a_2}^3a_3\ -4\,f_4f_7a_1{a_3}^3\ -5\,f_4f_7{a_2}^2{a_3}^2\ +12\,{f_5}^2a_0{a_3}^3\ -3\,{f_5}^2{a_2}^3a_3\ -6\,f_5f_6{a_2}^2{a_3}^2$
$-7\,f_5f_7a_2{a_3}^3.$

$\mathbf{eq}[\mathbf{12}]=-3\,a_0a_3{c_2}^2\ +a_1a_2c_1c_3\ +a_1a_2{c_2}^2\ +2\,a_1a_3c_0c_3\ -a_1a_3c_1c_2\ -2\,a_2a_3c_0c_2\ -a_2a_3{c_1}^2$
$-3\,{a_3}^2c_0c_1-4\,f_0{a_0}^2a_1c_2\ +4\,f_0{a_0}^2a_3c_0\ -2\,f_0a_0{a_1}^2c_1\ -8\,f_0a_0a_1a_2c_0\ +2\,f_0{a_1}^3c_0\ +2\,f_1{a_0}^2a_1c_3\ -f_1{a_0}^2a_3c_1$
$-3\,f_1a_0{a_1}^2c_2\ -5\,f_1a_0a_1a_3c_0\ -f_1{a_1}^3c_1\ +2\,f_2{a_0}^2a_2c_3\ +4\,f_2{a_0}^2a_3c_2\ -4\,f_2a_0a_1a_2c_2\ +3\,f_2a_0a_1a_3c_1$
$-2\,f_2a_0{a_2}^2c_1\ -f_2{a_1}^2a_2c_1\ -2\,f_2{a_1}^2a_3c_0\ +f_3{a_0}^2a_3c_3\ -2\,f_3a_0a_1a_2c_3\ -f_3a_0a_1a_3c_2\ +3\,f_3a_0{a_3}^2c_0\ +f_3{a_1}^2a_2c_2$
$+f_3{a_1}^2a_3c_1\ -f_3a_1a_2a_3c_0\ +2\,f_4a_0{a_2}^2c_3\ -4\,f_4a_0a_2a_3c_2\ -f_4a_0{a_3}^2c_1\ -f_4{a_1}^2a_2c_3\ -f_4{a_1}^2a_3c_2\ +f_4a_1{a_3}^2c_0$
$-2\,f_4{a_2}^3c_1\ -4\,f_4{a_2}^2a_3c_0\ +8\,f_5a_0a_2a_3c_3\ -14\,f_5a_0{a_3}^2c_2\ -2\,f_5{a_1}^2a_3c_3\ -f_5a_1{a_2}^2c_3\ -f_5a_1{a_3}^2c_1\ -5\,f_5{a_2}^2a_3c_1$
$-10\,f_5a_2{a_3}^2c_0\ +4\,f_6a_0{a_3}^2c_3\ -4\,f_6a_1a_2a_3c_3\ +2\,f_6{a_2}^2a_3c_2\ +4\,f_6a_2{a_3}^2c_1\ -4\,f_7a_1{a_3}^2c_3\ -f_7{a_2}^2a_3c_3$
$+4\,f_7a_2{a_3}^2c_2\ +8\,f_7{a_3}^3c_1-12\,f_0f_5{a_0}^2a_1a_3\ +8\,f_0f_5a_0{a_1}^2a_2\ -2\,f_0f_5{a_1}^4\ -8\,f_0f_6{a_0}^2a_2a_3\ +4\,f_0f_6a_0{a_1}^2a_3$
$+12\,f_0f_6a_0a_1{a_2}^2\ -4\,f_0f_6{a_1}^3a_2\ -12\,f_0f_7{a_0}^2{a_3}^2\ +24\,f_0f_7a_0a_1a_2a_3\ -4\,f_0f_7{a_1}^3a_3\ -2\,f_0f_7{a_1}^2{a_2}^2\ +{f_1}^2{a_0}^3a_1$
$+f_1f_2{a_0}^2{a_1}^2\ -2\,f_1f_3{a_0}^2a_1a_2\ +f_1f_3a_0{a_1}^3\ +2\,f_1f_4{a_0}^2a_1a_3\ -3\,f_1f_4a_0{a_1}^2a_2\ +f_1f_4{a_1}^4\ +2\,f_1f_5{a_0}^2a_2a_3$
$-5\,f_1f_5a_0{a_1}^2a_3\ +f_1f_5{a_1}^3a_2\ +8\,f_1f_6a_0a_1a_2a_3\ +16\,f_1f_7a_0a_1{a_3}^2\ -f_1f_7a_0{a_2}^2a_3\ -f_1f_7{a_1}^2a_2a_3-{f_2}^2{a_0}^3a_3$
$+{f_2}^2{a_0}^2a_1a_2\ -2\,f_2f_3{a_0}^2a_1a_3\ +f_2f_3a_0{a_1}^2a_2\ +2\,f_2f_4{a_0}^2a_2a_3\ -f_2f_4a_0{a_1}^2a_3\ -2\,f_2f_4a_0a_1{a_2}^2\ +f_2f_4{a_1}^3a_2$
$+10\,f_2f_5{a_0}^2{a_3}^2\ -14\,f_2f_5a_0a_1a_2a_3\ +2\,f_2f_5a_0{a_2}^3\ +2\,f_2f_5{a_1}^3a_3\ +f_2f_5{a_1}^2{a_2}^2\ -4\,f_2f_6a_0a_1{a_3}^2\ +2\,f_2f_6a_0{a_2}^2a_3$
$+4\,f_2f_6{a_1}^2a_2a_3\ +4\,f_2f_7{a_1}^2{a_3}^2\ +f_2f_7a_1{a_2}^2a_3\ -{f_3}^2a_0{a_1}^2a_3\ +{f_3}^2a_0a_1{a_2}^2\ -f_3f_4{a_1}^3a_3\ +f_3f_4{a_1}^2{a_2}^2$
$-4\,f_3f_5a_0a_1{a_3}^2\ -f_3f_5a_0{a_2}^2a_3\ +3\,f_3f_5{a_1}^2a_2a_3\ -6\,f_3f_6a_0a_2{a_3}^2\ +2\,f_3f_6a_1{a_2}^2a_3\ -9\,f_3f_7a_0{a_3}^3$
$+2\,f_3f_7a_1a_2{a_3}^2\ +f_3f_7{a_2}^3a_3\ +{f_4}^2a_0a_1{a_3}^2\ -{f_4}^2a_0{a_2}^2a_3\ -{f_4}^2{a_1}^2a_2a_3\ +{f_4}^2a_1{a_2}^3\ -8\,f_4f_5a_0a_2{a_3}^2$
$-3\,f_4f_5{a_1}^2{a_3}^2\ +2\,f_4f_5a_1{a_2}^2a_3\ +2\,f_4f_5{a_2}^4\ +6\,f_4f_6{a_2}^3a_3\ -2\,f_4f_7a_1{a_3}^3\ +11\,f_4f_7{a_2}^2{a_3}^2\ -16\,{f_5}^2a_0{a_3}^3$
$-4\,{f_5}^2a_1a_2{a_3}^2\ +6\,{f_5}^2{a_2}^3a_3\ +16\,f_5f_6{a_2}^2{a_3}^2\ +26\,f_5f_7a_2{a_3}^3.$

$\mathbf{eq}[\mathbf{13}]=-a_0a_1c_2c_3\ +a_0a_2c_1c_3\ +a_0a_3c_0c_3+4\,a_0a_3c_1c_2\ +{a_1}^2{c_2}^2-4\,f_0{a_0}^3c_2\ -4\,f_0{a_0}^2a_2c_0\ +2\,f_1{a_0}^2a_2c_1$
$-2\,f_1{a_0}^2a_3c_0\ -2\,f_3{a_0}^2a_2c_3\ -4\,f_3{a_0}^2a_3c_2\ +2\,f_3a_0a_1a_2c_2\ +2\,f_4{a_0}^2a_3c_3\ -2\,f_4a_0a_1a_2c_3\ -6\,f_4a_0a_1a_3c_2$
$-4\,f_4a_0{a_3}^2c_0\ +2\,f_4{a_1}^2a_2c_2\ -f_5a_0{a_2}^2c_3\ -4\,f_5a_0a_2a_3c_2\ +12\,f_5a_0{a_3}^2c_1\ +2\,f_5{a_1}^2a_3c_2\ -4\,f_5a_1a_2a_3c_1$
$-4\,f_5a_1{a_3}^2c_0\ -2\,f_6a_0a_2a_3c_3\ -4\,f_6a_0{a_3}^2c_2\ -5\,f_7a_0{a_3}^2c_3\ -2\,f_7a_1{a_3}^2c_2-12\,f_0f_5{a_0}^3a_3\ +4\,f_0f_5{a_0}^2a_1a_2$
$+4\,f_0f_6{a_0}^2{a_2}^2\ +16\,f_0f_7{a_0}^2a_2a_3\ -4\,f_0f_7a_0a_1{a_2}^2\ +{f_1}^2{a_0}^4\ -2\,f_1f_3{a_0}^3a_2\ +2\,f_1f_4{a_0}^3a_3\ -2\,f_1f_4{a_0}^2a_1a_2$
$+2\,f_1f_5{a_0}^2a_1a_3\ -2\,f_1f_5{a_0}^2{a_2}^2\ +8\,f_1f_7{a_0}^2{a_3}^2\ -4\,f_1f_7a_0a_1a_2a_3\ +{f_3}^2{a_0}^2{a_2}^2\ -2\,f_3f_4{a_0}^2a_2a_3\ +2\,f_3f_4a_0a_1{a_2}^2$
$-12\,f_3f_5{a_0}^2{a_3}^2\ +6\,f_3f_5a_0a_1a_2a_3\ +2\,f_3f_7a_0a_2{a_3}^2\ +{f_4}^2{a_0}^2{a_3}^2\ -2\,{f_4}^2a_0a_1a_2a_3\ +{f_4}^2{a_1}^2{a_2}^2\ -14\,f_4f_5a_0a_1{a_3}^2$
$+6\,f_4f_5{a_1}^2a_2a_3\ +4\,f_4f_6a_0a_2{a_3}^2\ +14\,f_4f_7a_0{a_3}^3\ -2\,f_4f_7a_1a_2{a_3}^2\ -12\,{f_5}^2a_0a_2{a_3}^2\ +{f_5}^2{a_1}^2{a_3}^2\ +4\,{f_5}^2a_1{a_2}^2a_3$

$-12\, f_5 f_6 a_0 a_3{}^3 \; +8\, f_5 f_6 a_1 a_2 a_3{}^2 \; +6\, f_5 f_7 a_1 a_3{}^3 \; +f_7{}^2 a_3{}^4.$

$\mathbf{eq[14]} = 18\, a_0 a_1 c_2 c_3 \; -16\, a_0 a_2 c_1 c_3 \; -50\, a_0 a_3 c_1 c_2 \; +4\, a_1{}^2 c_1 c_3 \; -14\, a_1{}^2 c_2{}^2 \; +2\, a_1 a_2 c_0 c_3 \; -4\, a_2{}^2 c_0 c_2 \; +3\, a_2{}^2 c_1{}^2$
$+a_3{}^2 c_0{}^2 + 44\, f_0 a_0{}^3 c_2 \; -8\, f_0 a_0{}^2 a_1 c_1 \; +28\, f_0 a_0{}^2 a_2 c_0 \; +8\, f_0 a_0 a_1{}^2 c_0 \; -4\, f_1 a_0{}^2 a_1 c_2 \; -24\, f_1 a_0{}^2 a_2 c_1 \; +10\, f_1 a_0{}^2 a_3 c_0$
$-2\, f_1 a_0 a_1{}^2 c_1 \; +6\, f_1 a_0 a_1 a_2 c_0 \; -2\, f_1 a_1{}^3 c_0 \; -6\, f_2 a_0{}^2 a_1 c_3 \; -4\, f_2 a_0{}^2 a_2 c_2 \; +6\, f_2 a_0{}^2 a_3 c_1$
$+2\, f_2 a_0 a_1{}^2 c_2 - 4\, f_2 a_0 a_1 a_3 c_0 \; +4\, f_2 a_0 a_2{}^2 c_0 \; -2\, f_2 a_1{}^2 a_2 c_0 \; +18\, f_3 a_0{}^2 a_2 c_3 \; +44\, f_3 a_0{}^2 a_3 c_2 \; -4\, f_3 a_0 a_1{}^2 c_3$
$-18\, f_3 a_0 a_1 a_2 c_2 \; -2\, f_3 a_0 a_1 a_3 c_1 \; +6\, f_3 a_0 a_2{}^2 c_1 + 2\, f_3 a_1{}^2 a_3 c_0 \; -2\, f_3 a_1 a_2{}^2 c_0 \; -30\, f_4 a_0{}^2 a_3 c_3 \; +30\, f_4 a_0 a_1 a_2 c_3$
$+70\, f_4 a_0 a_1 a_3 c_2 \; +4\, f_4 a_0 a_2{}^2 c_2 \; +40\, f_4 a_0 a_3{}^2 c_0 \; -4\, f_4 a_1{}^3 c_3 \; -26\, f_4 a_1{}^2 a_2 c_2 \; -4\, f_4 a_1{}^2 a_3 c_1 \; +4\, f_4 a_1 a_2 a_3 c_0$
$-4\, f_4 a_2{}^3 c_0 \; +18\, f_5 a_0 a_2{}^2 c_3 \; +52\, f_5 a_0 a_2 a_3 c_2 \; -144\, f_5 a_0 a_3{}^2 c_1 \; -6\, f_5 a_1{}^2 a_2 c_3 \; -26\, f_5 a_1{}^2 a_3 c_2 \; +54\, f_5 a_1 a_2 a_3 c_1$
$+54\, f_5 a_1 a_3{}^2 c_0 \; -6\, f_5 a_2{}^3 c_1 \; -12\, f_5 a_2{}^2 a_3 c_0 \; +12\, f_6 a_0 a_2 a_3 c_3 \; +44\, f_6 a_0 a_3{}^2 c_2 \; -4\, f_6 a_1 a_2{}^2 c_3 \; +4\, f_6 a_1 a_3{}^2 c_1$
$+4\, f_6 a_2{}^3 c_2 \; -8\, f_6 a_2{}^2 a_3 c_1 \; -8\, f_6 a_2 a_3{}^2 c_0 \; +4\, f_7 a_0 a_3{}^2 c_3 \; +30\, f_7 a_1 a_3{}^2 c_2 \; -2\, f_7 a_2{}^3 c_3 \; +14\, f_7 a_2{}^2 a_3 c_2$
$+10\, f_7 a_2 a_3{}^2 c_1 + 132\, f_0 f_5 a_0{}^3 a_3 \; -28\, f_0 f_5 a_0{}^2 a_1 a_2 \; -8\, f_0 f_5 a_0 a_1{}^3 \; -12\, f_0 f_6 a_0{}^2 a_2{}^2 \; -16\, f_0 f_6 a_0 a_1{}^2 a_2$
$-128\, f_0 f_7 a_0{}^2 a_2 a_3 - 32\, f_0 f_7 a_0 a_1{}^2 a_3 \; +52\, f_0 f_7 a_0 a_1 a_2{}^2 \; -11\, f_1{}^2 a_0{}^4 \; +4\, f_1 f_2 a_0{}^3 a_1 \; +22\, f_1 f_3 a_0{}^3 a_2 \; +4\, f_1 f_3 a_0{}^2 a_1{}^2$
$-22\, f_1 f_4 a_0{}^3 a_3 \; +18\, f_1 f_4 a_0{}^2 a_1 a_2 \; +4\, f_1 f_4 a_0 a_1{}^3 \; -30\, f_1 f_5 a_0{}^2 a_1 a_3 \; +26\, f_1 f_5 a_0{}^2 a_2{}^2 \; -2\, f_1 f_5 a_0 a_1{}^2 a_2 \; +2\, f_1 f_5 a_1{}^4$
$+24\, f_1 f_6 a_0{}^2 a_2 a_3 \; -4\, f_1 f_6 a_0 a_1{}^2 a_3 \; -8\, f_1 f_6 a_0 a_1 a_2{}^2 \; +4\, f_1 f_6 a_1{}^3 a_2 \; -52\, f_1 f_7 a_0{}^2 a_3{}^2 \; +32\, f_1 f_7 a_0 a_1 a_2 a_3$
$-2\, f_1 f_7 a_0 a_2{}^3 \; +4\, f_1 f_7 a_1{}^3 a_3 \; +2\, f_1 f_7 a_1{}^2 a_2{}^2 \; +4\, f_2{}^2 a_0{}^3 a_2 + 4\, f_2 f_3 a_0{}^2 a_1 a_2 \; +4\, f_2 f_4 a_0{}^2 a_1 a_3 \; -8\, f_2 f_4 a_0{}^2 a_2{}^2$
$+4\, f_2 f_4 a_0 a_1{}^2 a_2 \; -16\, f_2 f_5 a_0{}^2 a_2 a_3 \; +10\, f_2 f_5 a_0 a_1{}^2 a_3 \; -2\, f_2 f_5 a_0 a_1 a_2{}^2 \; +2\, f_2 f_5 a_1{}^3 a_2 \; +4\, f_2 f_6 a_0 a_1 a_2 a_3$
$-4\, f_2 f_6 a_0 a_2{}^3 \; +4\, f_2 f_6 a_1{}^2 a_2{}^2 \; +22\, f_2 f_7 a_0 a_1 a_3{}^2 \; -18\, f_2 f_7 a_0 a_2{}^2 a_3 \; +4\, f_2 f_7 a_1{}^2 a_2 a_3 \; +2\, f_2 f_7 a_1 a_2{}^3 \; -11\, f_3{}^2 a_0{}^2 a_2{}^2$
$+22\, f_3 f_4 a_0{}^2 a_2 a_3 \; +4\, f_3 f_4 a_0 a_1{}^2 a_3 \; -26\, f_3 f_4 a_0 a_1 a_2{}^2 \; +132\, f_3 f_5 a_0{}^2 a_3{}^2 \; -50\, f_3 f_5 a_0 a_1 a_2 a_3 \; -8\, f_3 f_5 a_0 a_2{}^3$
$-2\, f_3 f_5 a_1{}^3 a_3 \; +2\, f_3 f_5 a_1{}^2 a_2{}^2 \; -4\, f_3 f_6 a_1{}^2 a_2 a_3 \; +4\, f_3 f_6 a_1 a_2{}^3 \; -22\, f_3 f_7 a_0 a_2 a_3{}^2 \; -4\, f_3 f_7 a_1{}^2 a_3{}^2 \; +2\, f_3 f_7 a_1 a_2{}^2 a_3$
$+2\, f_3 f_7 a_2{}^4 \; -11\, f_4{}^2 a_0{}^2 a_3{}^2 \; +18\, f_4{}^2 a_0 a_1 a_2 a_3 \; +4\, f_4{}^2 a_0 a_2{}^3 \; +4\, f_4{}^2 a_1{}^3 a_3 \; -15\, f_4{}^2 a_1{}^2 a_2{}^2 \; +162\, f_4 f_5 a_0 a_1 a_3{}^2$
$+12\, f_4 f_5 a_0 a_2{}^2 a_3 \; -74\, f_4 f_5 a_1{}^2 a_2 a_3 \; -36\, f_4 f_6 a_0 a_2 a_3{}^2 \; -4\, f_4 f_6 a_1{}^2 a_3{}^2 \; +4\, f_4 f_6 a_2{}^4 \; -142\, f_4 f_7 a_0 a_3{}^3$
$+6\, f_4 f_7 a_1 a_2 a_3{}^2 \; +16\, f_4 f_7 a_2{}^3 a_3 \; +148\, f_5{}^2 a_0 a_2 a_3{}^2 \; -7\, f_5{}^2 a_1{}^2 a_3{}^2 \; -54\, f_5{}^2 a_1 a_2{}^2 a_3 \; +3\, f_5{}^2 a_2{}^4 \; +132\, f_5 f_6 a_0 a_3{}^3$
$-108\, f_5 f_6 a_1 a_2 a_3{}^2 \; +20\, f_5 f_6 a_2{}^3 a_3 \; -80\, f_5 f_7 a_1 a_3{}^3 \; +34\, f_5 f_7 a_2{}^2 a_3{}^2 \; +12\, f_6{}^2 a_2{}^2 a_3{}^2 \; +12\, f_6 f_7 a_2 a_3{}^3 \; -20\, f_7{}^2 a_3{}^4.$

$\mathbf{eq[15]} = -9\, a_0 a_1 c_2 c_3 \; +8\, a_0 a_2 c_1 c_3 \; +27\, a_0 a_3 c_1 c_2 \; -2\, a_1{}^2 c_1 c_3 \; +7\, a_1{}^2 c_2{}^2 \; -a_1 a_2 c_0 c_3 \; +2\, a_2{}^2 c_0 c_2 \; -a_2{}^2 c_1{}^2$
$+a_2 a_3 c_0 c_1 - 24\, f_0 a_0{}^3 c_2 \; +4\, f_0 a_0{}^2 a_1 c_1 \; -16\, f_0 a_0{}^2 a_2 c_0 \; -4\, f_0 a_0 a_1{}^2 c_0 \; +2\, f_1 a_0{}^2 a_1 c_2 \; +13\, f_1 a_0{}^2 a_2 c_1$
$-6\, f_1 a_0{}^2 a_3 c_0 + f_1 a_0 a_1{}^2 c_1 \; -3\, f_1 a_0 a_1 a_2 c_0 \; +f_1 a_1{}^3 c_0 \; +3\, f_2 a_0{}^2 a_1 c_3 \; +2\, f_2 a_0{}^2 a_2 c_2 \; -3\, f_2 a_0{}^2 a_3 c_1 \; -f_2 a_0 a_1{}^2 c_2$
$+2\, f_2 a_0 a_1 a_3 c_0 \; -2\, f_2 a_0 a_2{}^2 c_0 \; +f_2 a_1{}^2 a_2 c_0 \; -10\, f_3 a_0{}^2 a_2 c_3 \; -24\, f_3 a_0{}^2 a_3 c_2 \; +2\, f_3 a_0 a_1{}^2 c_3 \; +10\, f_3 a_0 a_1 a_2 c_2$
$+f_3 a_0 a_1 a_3 c_1 \; -3\, f_3 a_0 a_2{}^2 c_1 \; -f_3 a_1{}^2 a_3 c_0 \; +f_3 a_1 a_2{}^2 c_0 \; +16\, f_4 a_0{}^2 a_3 c_3 \; -16\, f_4 a_0 a_1 a_2 c_3 \; -38\, f_4 a_0 a_1 a_3 c_2$
$-2\, f_4 a_0 a_2{}^2 c_2 \; -22\, f_4 a_0 a_3{}^2 c_0 \; +2\, f_4 a_1{}^3 c_3 \; +14\, f_4 a_1{}^2 a_2 c_2 \; +2\, f_4 a_1{}^2 a_3 c_1 \; -2\, f_4 a_1 a_2 a_3 c_0 \; +2\, f_4 a_2{}^3 c_0$
$-9\, f_5 a_0 a_2{}^2 c_3 \; -28\, f_5 a_0 a_2 a_3 c_2 \; +78\, f_5 a_0 a_3{}^2 c_1 \; +3\, f_5 a_1{}^2 a_2 c_3 \; +13\, f_5 a_1{}^2 a_3 c_2 \; -28\, f_5 a_1 a_2 a_3 c_1 \; -28\, f_5 a_1 a_3{}^2 c_0$
$+2\, f_5 a_2{}^3 c_1 \; +5\, f_5 a_2{}^2 a_3 c_0 \; -6\, f_6 a_0 a_2 a_3 c_3 \; -24\, f_6 a_0 a_3{}^2 c_2 \; +2\, f_6 a_1 a_2{}^2 c_3 \; -2\, f_6 a_1 a_3{}^2 c_1 \; -2\, f_6 a_2{}^3 c_2 \; +2\, f_6 a_2{}^2 a_3 c_1$
$+2\, f_6 a_2 a_3{}^2 c_0 \; -6\, f_7 a_0 a_3{}^2 c_3 \; -13\, f_7 a_1 a_3{}^2 c_2 \; +f_7 a_2{}^3 c_3 \; -7\, f_7 a_2{}^2 a_3 c_2 \; -5\, f_7 a_2 a_3{}^2 c_1 - 72\, f_0 f_5 a_0{}^3 a_3$
$+16\, f_0 f_5 a_0{}^2 a_1 a_2 \; +4\, f_0 f_5 a_0 a_1{}^3 \; +8\, f_0 f_6 a_0{}^2 a_2{}^2 \; +8\, f_0 f_6 a_0 a_1{}^2 a_2 \; +72\, f_0 f_7 a_0{}^2 a_2 a_3 \; +16\, f_0 f_7 a_0 a_1{}^2 a_3$
$-28\, f_0 f_7 a_0 a_1 a_2{}^2 \; +6\, f_1{}^2 a_0{}^4 \; -2\, f_1 f_2 a_0{}^3 a_1 \; -12\, f_1 f_3 a_0{}^3 a_2 \; -2\, f_1 f_3 a_0{}^2 a_1{}^2 \; +12\, f_1 f_4 a_0{}^3 a_3 \; -10\, f_1 f_4 a_0{}^2 a_1 a_2$
$-2\, f_1 f_4 a_0 a_1{}^3 \; +16\, f_1 f_5 a_0{}^2 a_1 a_3 \; -14\, f_1 f_5 a_0{}^2 a_2{}^2 \; +f_1 f_5 a_0 a_1{}^2 a_2 \; -f_1 f_5 a_1{}^4 \; -12\, f_1 f_6 a_0{}^2 a_2 a_3 \; +2\, f_1 f_6 a_0 a_1{}^2 a_3$
$+4\, f_1 f_6 a_0 a_1 a_2{}^2 \; -2\, f_1 f_6 a_1{}^3 a_2 \; +30\, f_1 f_7 a_0{}^2 a_3{}^2 \; -18\, f_1 f_7 a_0 a_1 a_2 a_3 \; +f_1 f_7 a_0 a_2{}^3 \; -2\, f_1 f_7 a_1{}^3 a_3 \; -f_1 f_7 a_1{}^2 a_2{}^2$
$-2\, f_2{}^2 a_0{}^3 a_2 \; -2\, f_2 f_3 a_0{}^2 a_1 a_2 \; -2\, f_2 f_4 a_0{}^2 a_1 a_3 \; +4\, f_2 f_4 a_0{}^2 a_2{}^2 \; -2\, f_2 f_4 a_0 a_1{}^2 a_2 \; +8\, f_2 f_5 a_0{}^2 a_2 a_3 \; -5\, f_2 f_5 a_0 a_1{}^2 a_3$
$+f_2 f_5 a_0 a_1 a_2{}^2 \; -f_2 f_5 a_1{}^3 a_2 \; -2\, f_2 f_6 a_0 a_1 a_2 a_3 \; +2\, f_2 f_6 a_0 a_2{}^3 \; -2\, f_2 f_6 a_1{}^2 a_2{}^2 \; -11\, f_2 f_7 a_0 a_1 a_3{}^2 \; +9\, f_2 f_7 a_0 a_2{}^2 a_3$
$-2\, f_2 f_7 a_1{}^2 a_2 a_3 \; -f_2 f_7 a_1 a_2{}^3 \; +6\, f_3{}^2 a_0{}^2 a_2{}^2 \; -12\, f_3 f_4 a_0{}^2 a_2 a_3 \; -2\, f_3 f_4 a_0 a_1{}^2 a_3 \; +14\, f_3 f_4 a_0 a_1 a_2{}^2$
$-72\, f_3 f_5 a_0{}^2 a_3{}^2 \; +28\, f_3 f_5 a_0 a_1 a_2 a_3 \; +4\, f_3 f_5 a_0 a_2{}^3 \; +f_3 f_5 a_1{}^3 a_3 \; -f_3 f_5 a_1{}^2 a_2{}^2 \; +2\, f_3 f_6 a_1{}^2 a_2 a_3 \; -2\, f_3 f_6 a_1 a_2{}^3$
$+12\, f_3 f_7 a_0 a_2 a_3{}^2 \; +2\, f_3 f_7 a_1{}^2 a_3{}^2 \; -f_3 f_7 a_1 a_2{}^2 a_3 \; -f_3 f_7 a_2{}^4 \; +6\, f_4{}^2 a_0{}^2 a_3{}^2 \; -10\, f_4{}^2 a_0 a_1 a_2 a_3 \; -2\, f_4{}^2 a_0 a_2{}^3$
$-2\, f_4{}^2 a_1{}^3 a_3 \; +8\, f_4{}^2 a_1{}^2 a_2{}^2 \; -88\, f_4 f_5 a_0 a_1 a_3{}^2 \; -6\, f_4 f_5 a_0 a_2{}^2 a_3 \; +40\, f_4 f_5 a_1{}^2 a_2 a_3 \; +20\, f_4 f_6 a_0 a_2 a_3{}^2$
$+2\, f_4 f_6 a_1{}^2 a_3{}^2 \; -2\, f_4 f_6 a_2{}^4 \; +78\, f_4 f_7 a_0 a_3{}^3 \; -4\, f_4 f_7 a_1 a_2 a_3{}^2 \; -8\, f_4 f_7 a_2{}^3 a_3 \; -80\, f_5{}^2 a_0 a_2 a_3{}^2 \; +4\, f_5{}^2 a_1{}^2 a_3{}^2$
$+28\, f_5{}^2 a_1 a_2{}^2 a_3 \; -f_5{}^2 a_2{}^4 \; -72\, f_5 f_6 a_0 a_3{}^3 \; +56\, f_5 f_6 a_1 a_2 a_3{}^2 - 8\, f_5 f_6 a_2{}^3 a_3 \; +46\, f_5 f_7 a_1 a_3{}^3 \; -17\, f_5 f_7 a_2{}^2 a_3{}^2$
$-4\, f_6{}^2 a_2{}^2 a_3{}^2 \; -6\, f_6 f_7 a_2 a_3{}^3 \; +6\, f_7{}^2 a_3{}^4.$

$\mathbf{eq[16]} = 9\, a_0 a_1 c_2 c_3 \; -9\, a_0 a_2 c_1 c_3 \; -27\, a_0 a_3 c_1 c_2 \; +2\, a_1{}^2 c_1 c_3 \; -7\, a_1{}^2 c_2{}^2 \; +a_1 a_2 c_0 c_3 \; +a_1 a_2 c_1 c_2 \; -2\, a_2{}^2 c_0 c_2$
$+2\, a_2{}^2 c_1{}^2 + 24\, f_0 a_0{}^3 c_2 \; -4\, f_0 a_0{}^2 a_1 c_1 \; +16\, f_0 a_0{}^2 a_2 c_0 \; +4\, f_0 a_0 a_1{}^2 c_0 \; -2\, f_1 a_0{}^2 a_1 c_2 \; -13\, f_1 a_0{}^2 a_2 c_1$
$+6\, f_1 a_0{}^2 a_3 c_0 - f_1 a_0 a_1{}^2 c_1 \; +3\, f_1 a_0 a_1 a_2 c_0 \; -f_1 a_1{}^3 c_0 \; -3\, f_2 a_0{}^2 a_1 c_3 \; -2\, f_2 a_0{}^2 a_2 c_2 \; +3\, f_2 a_0{}^2 a_3 c_1 \; +f_2 a_0 a_1{}^2 c_2$
$-2\, f_2 a_0 a_1 a_3 c_0 \; +2\, f_2 a_0 a_2{}^2 c_0 \; -f_2 a_1{}^2 a_2 c_0 \; +10\, f_3 a_0{}^2 a_2 c_3 \; +24\, f_3 a_0{}^2 a_3 c_2 \; -2\, f_3 a_0 a_1{}^2 c_3 \; -10\, f_3 a_0 a_1 a_2 c_2$

$-f_3a_0a_1a_3c_1 +3\,f_3a_0a_2{}^2c_1 +f_3a_1{}^2a_3c_0 -f_3a_1a_2{}^2c_0 -16\,f_4a_0{}^2a_3c_3 +16\,f_4a_0a_1a_2c_3 +38\,f_4a_0a_1a_3c_2$
$+2\,f_4a_0a_2{}^2c_2 +22\,f_4a_0a_3{}^2c_0 -2\,f_4a_1{}^3c_3 -14\,f_4a_1{}^2a_2c_2 -2\,f_4a_1{}^2a_3c_1 +2\,f_4a_1a_2a_3c_0 -2\,f_4a_2{}^3c_0$
$+9\,f_5a_0a_2{}^2c_3 +28\,f_5a_0a_2a_3c_2 -78\,f_5a_0a_3{}^2c_1 -3\,f_5a_1{}^2a_2c_3 -13\,f_5a_1{}^2a_3c_2 +30\,f_5a_1a_2a_3c_1 +28\,f_5a_1a_3{}^2c_0$
$-3\,f_5a_2{}^3c_1 -5\,f_5a_2{}^2a_3c_0 +6\,f_6a_0a_2a_3c_3 +24\,f_6a_0a_3{}^2c_2 -2\,f_6a_1a_2{}^2c_3 +2\,f_6a_1a_3{}^2c_1 +2\,f_6a_2{}^3c_2 -4\,f_6a_2{}^2a_3c_1$
$-2\,f_6a_2a_3{}^2c_0 +6\,f_7a_0a_3{}^2c_3 +13\,f_7a_1a_3{}^2c_2 -f_7a_2{}^3c_3 +7\,f_7a_2{}^2a_3c_2 +2\,f_7a_2a_3{}^2c_1 + 72\,f_0f_5a_0{}^3a_3$
$-16\,f_0f_5a_0{}^2a_1a_2 -4\,f_0f_5a_0a_1{}^3 -8\,f_0f_6a_0{}^2a_2{}^2 -8\,f_0f_6a_0a_1{}^2a_2 -72\,f_0f_7a_0{}^2a_2a_3 -16\,f_0f_7a_0a_1{}^2a_3$
$+28\,f_0f_7a_0a_1a_2{}^2 -6\,f_1{}^2a_0{}^4 +2\,f_1f_2a_0{}^3a_1 +12\,f_1f_3a_0{}^3a_2 +2\,f_1f_3a_0{}^2a_1{}^2 -12\,f_1f_4a_0{}^3a_3 +10\,f_1f_4a_0{}^2a_1a_2$
$+2\,f_1f_4a_0a_1{}^3 -16\,f_1f_5a_0{}^2a_1a_3 +14\,f_1f_5a_0{}^2a_2{}^2 -f_1f_5a_0a_1{}^2a_2 +f_1f_5a_1{}^4 +12\,f_1f_6a_0{}^2a_2a_3 -2\,f_1f_6a_0a_1{}^2a_3$
$-4\,f_1f_6a_0a_1a_2{}^2 +2\,f_1f_6a_1{}^3a_2 -30\,f_1f_7a_0{}^2a_3{}^2 +18\,f_1f_7a_0a_1a_2a_3 -f_1f_7a_0a_2{}^3 +2\,f_1f_7a_1{}^3a_3 +f_1f_7a_1{}^2a_2{}^2$
$+2\,f_2{}^2a_0{}^3a_2 +2\,f_2f_3a_0{}^2a_1a_2 +2\,f_2f_4a_0{}^2a_1a_3 -4\,f_2f_4a_0{}^2a_2{}^2 +2\,f_2f_4a_0a_1{}^2a_2 -8\,f_2f_5a_0{}^2a_2a_3 +5\,f_2f_5a_0a_1{}^2a_3$
$-f_2f_5a_0a_1a_2{}^2 +f_2f_5a_1{}^3a_2 +2\,f_2f_6a_0a_1a_2a_3 -2\,f_2f_6a_0a_2{}^3 +2\,f_2f_6a_1{}^2a_2{}^2 +11\,f_2f_7a_0a_1a_3{}^2 -9\,f_2f_7a_0a_2{}^2a_3$
$+2\,f_2f_7a_1{}^2a_2a_3 +f_2f_7a_1a_2{}^3 -6\,f_3{}^2a_0{}^2a_2{}^2 +12\,f_3f_4a_0{}^2a_2a_3 +2\,f_3f_4a_0a_1{}^2a_3 -14\,f_3f_4a_0a_1a_2{}^2$
$+72\,f_3f_5a_0{}^2a_3{}^2 -28\,f_3f_5a_0a_1a_2a_3 -4\,f_3f_5a_0a_2{}^3 -f_3f_5a_1{}^3a_3 +f_3f_5a_1{}^2a_2{}^2 -2\,f_3f_6a_1{}^2a_2a_3 +2\,f_3f_6a_1a_2{}^3$
$-12\,f_3f_7a_0a_2a_3{}^2 -2\,f_3f_7a_1{}^2a_3{}^2 +f_3f_7a_1a_2{}^2a_3 +f_3f_7a_2{}^4 -6\,f_4{}^2a_0{}^2a_3{}^2 +10\,f_4{}^2a_0a_1a_2a_3 +2\,f_4{}^2a_0a_2{}^3$
$+2\,f_4{}^2a_1{}^3a_3 -8\,f_4{}^2a_1{}^2a_2{}^2 +88\,f_4f_5a_0a_1a_3{}^2 +6\,f_4f_5a_0a_2{}^2a_3 -40\,f_4f_5a_1{}^2a_2a_3 -20\,f_4f_6a_0a_2a_3{}^2$
$-2\,f_4f_6a_1{}^2a_3{}^2 +2\,f_4f_6a_2{}^4 -78\,f_4f_7a_0a_3{}^3 +4\,f_4f_7a_1a_2a_3{}^2 +8\,f_4f_7a_2{}^3a_3 +80\,f_5{}^2a_0a_2a_3{}^2 -4\,f_5{}^2a_1{}^2a_3{}^2$
$-28\,f_5{}^2a_1a_2{}^2a_3 +f_5{}^2a_2{}^4 +72\,f_5f_6a_0a_3{}^3 -56\,f_5f_6a_1a_2a_3{}^2 + 8\,f_5f_6a_2{}^3a_3 -46\,f_5f_7a_1a_3{}^3 +17\,f_5f_7a_2{}^2a_3{}^2$
$+4\,f_6{}^2a_2{}^2a_3{}^2 +6\,f_6f_7a_2a_3{}^3 -6\,f_7{}^2a_3{}^4.$

$\mathbf{eq}[\mathbf{17}] = -10\,a_0a_1c_2c_3 +9\,a_0a_2c_1c_3 +27\,a_0a_3c_1c_2 -2\,a_1{}^2c_1c_3 +8\,a_1{}^2c_2{}^2 -a_1a_2c_0c_3 +a_1a_3c_0c_2 +2\,a_2{}^2c_0c_2$
$-2\,a_2{}^2c_1{}^2 - 24\,f_0a_0{}^3c_2 +4\,f_0a_0{}^2a_1c_1 -16\,f_0a_0{}^2a_2c_0 -4\,f_0a_0a_1{}^2c_0 +2\,f_1a_0{}^2a_1c_2 +13\,f_1a_0{}^2a_2c_1$
$-6\,f_1a_0{}^2a_3c_0 + f_1a_0a_1{}^2c_1 -3\,f_1a_0a_1a_2c_0 +f_1a_1{}^3c_0 +3\,f_2a_0{}^2a_1c_3 +2\,f_2a_0{}^2a_2c_2 -3\,f_2a_0{}^2a_3c_1 -f_2a_0a_1{}^2c_2$
$+2\,f_2a_0a_1a_3c_0 -2\,f_2a_0a_2{}^2c_0 +f_2a_1{}^2a_2c_0 -10\,f_3a_0{}^2a_2c_3 -24\,f_3a_0{}^2a_3c_2 +2\,f_3a_0a_1{}^2c_3 +10\,f_3a_0a_1a_2c_2$
$+f_3a_0a_1a_3c_1 -3\,f_3a_0a_2{}^2c_1 -f_3a_1{}^2a_3c_0 +f_3a_1a_2{}^2c_0 +16\,f_4a_0{}^2a_3c_3 -16\,f_4a_0a_1a_2c_3 -38\,f_4a_0a_1a_3c_2$
$-2\,f_4a_0a_2{}^2c_2 -22\,f_4a_0a_3{}^2c_0 +2\,f_4a_1{}^3c_3 +14\,f_4a_1{}^2a_2c_2 +2\,f_4a_1{}^2a_3c_1 -2\,f_4a_1a_2a_3c_0 +2\,f_4a_2{}^3c_0$
$-10\,f_5a_0a_2{}^2c_3 -28\,f_5a_0a_2a_3c_2 +78\,f_5a_0a_3{}^2c_1 +3\,f_5a_1{}^2a_2c_3 +15\,f_5a_1{}^2a_3c_2 -30\,f_5a_1a_2a_3c_1 -28\,f_5a_1a_3{}^2c_0$
$+4\,f_5a_2{}^3c_1 +6\,f_5a_2{}^2a_3c_0 -8\,f_6a_0a_2a_3c_3 -24\,f_6a_0a_3{}^2c_2 +2\,f_6a_1a_2{}^2c_3 -2\,f_6a_1a_3{}^2c_1 -2\,f_6a_2{}^3c_2 +6\,f_6a_2{}^2a_3c_1$
$+4\,f_6a_2a_3{}^2c_0 -6\,f_7a_0a_3{}^2c_3 -16\,f_7a_1a_3{}^2c_2 +f_7a_2{}^3c_3 -7\,f_7a_2{}^2a_3c_2 -2\,f_7a_2a_3{}^2c_1 - 72\,f_0f_5a_0{}^3a_3$
$+16\,f_0f_5a_0{}^2a_1a_2 +4\,f_0f_5a_0a_1{}^3 +8\,f_0f_6a_0{}^2a_2{}^2 +8\,f_0f_6a_0a_1{}^2a_2 +72\,f_0f_7a_0{}^2a_2a_3 +16\,f_0f_7a_0a_1{}^2a_3$
$-28\,f_0f_7a_0a_1a_2{}^2 +6\,f_1{}^2a_0{}^4 -2\,f_1f_2a_0{}^3a_1 -12\,f_1f_3a_0{}^3a_2 -2\,f_1f_3a_0{}^2a_1{}^2 +12\,f_1f_4a_0{}^3a_3 -10\,f_1f_4a_0{}^2a_1a_2$
$-2\,f_1f_4a_0a_1{}^3 +16\,f_1f_5a_0{}^2a_1a_3 -14\,f_1f_5a_0{}^2a_2{}^2 +f_1f_5a_0a_1{}^2a_2 -f_1f_5a_1{}^4 -12\,f_1f_6a_0{}^2a_2a_3 +2\,f_1f_6a_0a_1{}^2a_3$
$+4\,f_1f_6a_0a_1a_2{}^2 -2\,f_1f_6a_1{}^3a_2 +30\,f_1f_7a_0{}^2a_3{}^2 -18\,f_1f_7a_0a_1a_2a_3 +f_1f_7a_0a_2{}^3 -2\,f_1f_7a_1{}^3a_3 -f_1f_7a_1{}^2a_2{}^2$
$-2\,f_2{}^2a_0{}^3a_2 -2\,f_2f_3a_0{}^2a_1a_2 -2\,f_2f_4a_0{}^2a_1a_3 +4\,f_2f_4a_0{}^2a_2{}^2 -2\,f_2f_4a_0a_1{}^2a_2 +8\,f_2f_5a_0{}^2a_2a_3 -5\,f_2f_5a_0a_1{}^2a_3$
$+f_2f_5a_0a_1a_2{}^2 -f_2f_5a_1{}^3a_2 -2\,f_2f_6a_0a_1a_2a_3 +2\,f_2f_6a_0a_2{}^3 -2\,f_2f_6a_1{}^2a_2{}^2 -11\,f_2f_7a_0a_1a_3{}^2 +9\,f_2f_7a_0a_2{}^2a_3$
$-2\,f_2f_7a_1{}^2a_2a_3 -f_2f_7a_1a_2{}^3 +6\,f_3{}^2a_0{}^2a_2{}^2 -12\,f_3f_4a_0{}^2a_2a_3 -2\,f_3f_4a_0a_1{}^2a_3 +14\,f_3f_4a_0a_1a_2{}^2$
$-72\,f_3f_5a_0{}^2a_3{}^2 +28\,f_3f_5a_0a_1a_2a_3 +4\,f_3f_5a_0a_2{}^3 +f_3f_5a_1{}^3a_3 -f_3f_5a_1{}^2a_2{}^2 +2\,f_3f_6a_1{}^2a_2a_3 -2\,f_3f_6a_1a_2{}^3$
$+12\,f_3f_7a_0a_2a_3{}^2 +2\,f_3f_7a_1{}^2a_3{}^2 -f_3f_7a_1a_2{}^2a_3 -f_3f_7a_2{}^4 +6\,f_4{}^2a_0{}^2a_3{}^2 -10\,f_4{}^2a_0a_1a_2a_3 -2\,f_4{}^2a_0a_2{}^3$
$-2\,f_4{}^2a_1{}^3a_3 +8\,f_4{}^2a_1{}^2a_2{}^2 -88\,f_4f_5a_0a_1a_3{}^2 -6\,f_4f_5a_0a_2{}^2a_3 +40\,f_4f_5a_1{}^2a_2a_3 +20\,f_4f_6a_0a_2a_3{}^2$
$+2\,f_4f_6a_1{}^2a_3{}^2 -2\,f_4f_6a_2{}^4 +78\,f_4f_7a_0a_3{}^3 -4\,f_4f_7a_1a_2a_3{}^2 -8\,f_4f_7a_2{}^3a_3 -80\,f_5{}^2a_0a_2a_3{}^2 +4\,f_5{}^2a_1{}^2a_3{}^2$
$+30\,f_5{}^2a_1a_2{}^2a_3 -2\,f_5{}^2a_2{}^4 -72\,f_5f_6a_0a_3{}^3 +60\,f_5f_6a_1a_2a_3{}^2 -12\,f_5f_6a_2{}^3a_3 +46\,f_5f_7a_1a_3{}^3 -20\,f_5f_7a_2{}^2a_3{}^2$
$-8\,f_6{}^2a_2{}^2a_3{}^2 -12\,f_6f_7a_2a_3{}^3 +6\,f_7{}^2a_3{}^4.$

$\mathbf{eq}[\mathbf{18}] = a_0{}^2c_3{}^2 -2\,a_0a_1c_2c_3 +4\,a_0a_3c_1c_2 +a_1{}^2c_2{}^2 - 4\,f_0a_0{}^3c_2 -4\,f_0a_0{}^2a_2c_0 +2\,f_1a_0{}^2a_2c_1 -2\,f_1a_0{}^2a_3c_0$
$-2\,f_3a_0{}^2a_2c_3 -4\,f_3a_0{}^2a_3c_2 +2\,f_3a_0a_1a_2c_2 +2\,f_4a_0{}^2a_3c_3 -2\,f_4a_0a_1a_2c_3 -6\,f_4a_0a_1a_3c_2 -4\,f_4a_0a_3{}^2c_0$
$+2\,f_4a_1{}^2a_2c_2 -4\,f_5a_0a_2a_3c_2 +12\,f_5a_0a_3{}^2c_1 -6\,f_5a_1a_2a_3c_1 -6\,f_5a_1a_3{}^2c_0 -4\,f_6a_0a_3{}^2c_2 -2\,f_7a_0a_3{}^2c_3$
$-2\,f_7a_1a_3{}^2c_2 - 12\,f_0f_5a_0{}^3a_3 +4\,f_0f_5a_0{}^2a_1a_2 +4\,f_0f_6a_0{}^2a_2{}^2 +16\,f_0f_7a_0{}^2a_2a_3 -4\,f_0f_7a_0a_1a_2{}^2 +f_1{}^2a_0{}^4$
$-2\,f_1f_3a_0{}^3a_2 +2\,f_1f_4a_0{}^3a_3 -2\,f_1f_4a_0{}^2a_1a_2 +2\,f_1f_5a_0{}^2a_1a_3 -2\,f_1f_5a_0{}^2a_2{}^2 +8\,f_1f_7a_0{}^2a_3{}^2 -4\,f_1f_7a_0a_1a_2a_3$
$+f_3{}^2a_0{}^2a_2{}^2 -2\,f_3f_4a_0{}^2a_2a_3 +2\,f_3f_4a_0a_1a_2{}^2 -12\,f_3f_5a_0{}^2a_3{}^2 +6\,f_3f_5a_0a_1a_2a_3 +2\,f_3f_7a_0a_2a_3{}^2 +f_4{}^2a_0{}^2a_3{}^2$
$-2\,f_4{}^2a_0a_1a_2a_3 +f_4{}^2a_1{}^2a_2{}^2 -14\,f_4f_5a_0a_1a_3{}^2 +6\,f_4f_5a_1{}^2a_2a_3 +4\,f_4f_6a_0a_2a_3{}^2 +14\,f_4f_7a_0a_3{}^3$
$-2\,f_4f_7a_1a_2a_3{}^2 -12\,f_5{}^2a_0a_2a_3{}^2 -3\,f_5{}^2a_1{}^2a_3{}^2 +6\,f_5{}^2a_1a_2{}^2a_3 -12\,f_5f_6a_0a_3{}^3 +12\,f_5f_6a_1a_2a_3{}^2 +12\,f_5f_7a_1a_3{}^3$
$+f_7{}^2a_3{}^4.$

$\mathbf{eq[19]} = -7\, a_0a_1c_2c_3 \ +7\, a_0a_2c_1c_3 \ +a_0a_2{c_2}^2 \ +18\, a_0a_3c_1c_2 \ -2\, {a_1}^2c_1c_3 \ +5\, {a_1}^2{c_2}^2 \ -a_1a_2c_0c_3 \ +a_1a_3{c_1}^2$
$+2\, {a_2}^2c_0c_2 \ -2\, {a_2}^2{c_1}^2 - 16\, f_0{a_0}^3c_2 \ +2\, f_0{a_0}^2a_1c_1 \ -12\, f_0{a_0}^2a_2c_0 \ -2\, f_0a_0{a_1}^2c_0 \ +f_1{a_0}^2a_1c_2 \ +9\, f_1{a_0}^2a_2c_1$
$-4\, f_1{a_0}^2a_3c_0 \ -3\, f_1a_0a_1a_2c_0 \ +f_1{a_1}^3c_0 \ +2\, f_2{a_0}^2a_1c_3 \ -2\, f_2{a_0}^2a_3c_1 \ +f_2a_0a_1a_3c_0 \ -2\, f_2a_0{a_2}^2c_0 \ +f_2{a_1}^2a_2c_0$
$-8\, f_3{a_0}^2a_2c_3 \ -16\, f_3{a_0}^2a_3c_2 \ +2\, f_3a_0{a_1}^2c_3 \ +7\, f_3a_0a_1a_2c_2 \ -f_3a_0{a_2}^2c_1 \ -f_3{a_1}^2a_3c_0 \ +f_3a_1{a_2}^2c_0$
$+11\, f_4{a_0}^2a_3c_3 \ -11\, f_4a_0a_1a_2c_3 \ -26\, f_4a_0a_1a_3c_2 \ -15\, f_4a_0{a_3}^2c_0 \ +2\, f_4{a_1}^3c_3 + 8\, f_4{a_1}^2a_2c_2 \ -3\, f_4a_1a_2a_3c_0$
$+2\, f_4{a_2}^3c_0 \ -8\, f_5a_0{a_2}^2c_3 \ -14\, f_5a_0a_2a_3c_2 \ +52\, f_5a_0{a_3}^2c_1 \ +3\, f_5{a_1}^2a_2c_3 \ +7\, f_5{a_1}^2a_3c_2 \ -23\, f_5a_1a_2a_3c_1$
$-22\, f_5a_1{a_3}^2c_0 \ +4\, f_5{a_2}^3c_1 \ +6\, f_5{a_2}^2a_3c_0 \ -6\, f_6a_0a_2a_3c_3 \ -16\, f_6a_0{a_3}^2c_2 \ +2\, f_6a_1{a_2}^2c_3 \ -2\, f_6a_1{a_3}^2c_1$
$-2\, f_6{a_2}^3c_2 \ +4\, f_6{a_2}^2a_3c_1 \ +2\, f_6a_2{a_3}^2c_0 \ -4\, f_7a_0{a_3}^2c_3 \ -9\, f_7a_1{a_3}^2c_2 \ +f_7{a_2}^3c_3 \ -6\, f_7{a_2}^2a_3c_2$
$-f_7a_2{a_3}^2c_1 - 48\, f_0f_5{a_0}^3a_3 \ +12\, f_0f_5{a_0}^2a_1a_2 \ +2\, f_0f_5a_0{a_1}^3 \ +8\, f_0f_6{a_0}^2{a_2}^2 \ +4\, f_0f_6a_0{a_1}^2a_2 \ +52\, f_0f_7{a_0}^2a_2a_3$
$+8\, f_0f_7a_0{a_1}^2a_3 \ -18\, f_0f_7a_0a_1{a_2}^2 \ +4\, {f_1}^2{a_0}^4 \ -f_1f_2{a_0}^3a_1 \ -8\, f_1f_3{a_0}^3a_2 \ -f_1f_3{a_0}^2{a_1}^2 \ +8\, f_1f_4{a_0}^3a_3$
$-7\, f_1f_4{a_0}^2a_1a_2 \ -f_1f_4a_0{a_1}^3 \ +10\, f_1f_5{a_0}^2a_1a_3 \ -10\, f_1f_5{a_0}^2{a_2}^2 \ +2\, f_1f_5a_0{a_1}^2a_2 \ -f_1f_5{a_1}^4 \ -8\, f_1f_6{a_0}^2a_2a_3$
$+2\, f_1f_6a_0{a_1}^2a_3 \ +4\, f_1f_6a_0a_1{a_2}^2 \ -2\, f_1f_6{a_1}^3a_2 \ +20\, f_1f_7{a_0}^2{a_3}^2 \ -9\, f_1f_7a_0a_1a_2a_3 \ +f_1f_7a_0{a_2}^3 \ -2\, f_1f_7{a_1}^3a_3$
$-f_1f_7{a_1}^2{a_2}^2 \ -{f_2}^2{a_0}^3a_2 \ -f_2f_3{a_0}^2a_1a_2 \ -f_2f_4{a_0}^2a_1a_3 \ +2\, f_2f_4{a_0}^2{a_2}^2 \ -f_2f_4a_0{a_1}^2a_2 \ +2\, f_2f_5{a_0}^2a_2a_3$
$-f_2f_5a_0{a_1}^2a_3 \ +f_2f_5a_0a_1{a_2}^2 \ -f_2f_5{a_1}^3a_2 \ +2\, f_2f_6a_0{a_2}^3 \ -2\, f_2f_6{a_1}^2{a_2}^2 \ -6\, f_2f_7a_0a_1{a_3}^2 \ +8\, f_2f_7a_0{a_2}^2a_3$
$-2\, f_2f_7{a_1}^2a_2a_3 \ -f_2f_7a_1{a_2}^3 \ +4\, {f_3}^2{a_0}^2{a_2}^2 \ -8\, f_3f_4{a_0}^2a_2a_3 \ -f_3f_4a_0{a_1}^2a_3 \ +9\, f_3f_4a_0a_1{a_2}^2 \ -48\, f_3f_5{a_0}^2{a_3}^2$
$+20\, f_3f_5a_0a_1a_2a_3 \ +2\, f_3f_5a_0{a_2}^3 \ +f_3f_5{a_1}^3a_3 \ -f_3f_5{a_1}^2{a_2}^2 \ +2\, f_3f_6{a_1}^2a_2a_3 \ -2\, f_3f_6a_1{a_2}^3 \ +8\, f_3f_7a_0a_2{a_3}^2$
$+2\, f_3f_7{a_1}^2{a_3}^2 \ -f_3f_7a_1{a_2}^2a_3 \ -f_3f_7{a_2}^4 \ +4\, {f_4}^2{a_0}^2{a_3}^2 \ -7\, {f_4}^2a_0a_1a_2a_3 \ -{f_4}^2a_0{a_2}^3 \ -{f_4}^2{a_1}^3a_3 \ +5\, {f_4}^2{a_1}^2{a_2}^2$
$-60\, f_4f_5a_0a_1{a_3}^2 \ -f_4f_5a_0{a_2}^2a_3 \ +25\, f_4f_5{a_1}^2a_2a_3 \ +14\, f_4f_6a_0a_2{a_3}^2 \ +2\, f_4f_6{a_1}^2{a_3}^2 \ +2\, f_4f_6a_1{a_2}^2a_3$
$-2\, f_4f_6{a_2}^4 \ +53\, f_4f_7a_0{a_3}^3 \ +2\, f_4f_7a_1a_2{a_3}^2 \ -7\, f_4f_7{a_2}^3a_3 \ -48\, {f_5}^2a_0a_2{a_3}^2 \ -2\, {f_5}^2{a_1}^2{a_3}^2 \ +22\, {f_5}^2a_1{a_2}^2a_3$
$-2\, {f_5}^2{a_2}^4 \ -48\, f_5f_6a_0{a_3}^3 \ +44\, f_5f_6a_1a_2{a_3}^2 \ -10\, f_5f_6{a_2}^3a_3 \ +40\, f_5f_7a_1{a_3}^3 \ -18\, f_5f_7{a_2}^2{a_3}^2 \ -4\, {f_6}^2{a_2}^2{a_3}^2$
$-6\, f_6f_7a_2{a_3}^3 \ +4\, {f_7}^2{a_3}^4.$

$\mathbf{eq[20]} = 2\, a_0a_1{c_2}^2 \ +a_0a_2c_0c_3 \ +5\, a_0a_3c_0c_2 \ +a_0a_3{c_1}^2 \ +2\, {a_1}^2c_1c_2 \ +a_1a_2{c_1}^2 \ +{a_2}^2c_0c_1 \ +a_2a_3{c_0}^2 - 2\, f_0{a_0}^3c_1$
$+2\, f_0{a_0}^2a_1c_0 \ -f_1{a_0}^3c_2 \ +2\, f_1{a_0}^2a_2c_0 \ -f_1a_0{a_1}^2c_0 \ -2\, f_2{a_0}^2a_1c_2 \ -f_2{a_0}^2a_3c_0 \ -f_2a_0a_1a_2c_0 \ +f_3{a_0}^2a_2c_2$
$-2\, f_3{a_0}^2a_3c_1 \ -2\, f_3a_0{a_1}^2c_2 \ +f_3a_0a_1a_3c_0 \ -2\, f_3a_0{a_2}^2c_0 \ +3\, f_4{a_0}^2a_2c_3 \ -f_4{a_0}^2a_3c_2 \ -2\, f_4a_0{a_1}^2c_3$
$-2\, f_4a_0a_1a_3c_1 \ -3\, f_4a_0{a_2}^2c_1 \ +2\, f_4{a_1}^2a_2c_1 \ +2\, f_4{a_1}^2a_3c_0 \ -f_4a_1{a_2}^2c_0 \ -2\, f_5{a_0}^2a_3c_3 \ -4\, f_5a_0a_1a_2c_3$
$+11\, f_5a_0a_1a_3c_2 \ -f_5a_0{a_2}^2c_2 \ +16\, f_5a_0{a_3}^2c_0 \ +f_5{a_1}^2a_2c_2 \ +6\, f_5{a_1}^2a_3c_1 \ +f_5a_1a_2a_3c_0 \ -f_5{a_2}^3c_0 \ -2\, f_6a_0{a_2}^2c_3$
$-8\, f_6a_0a_2a_3c_2 \ -2\, f_6a_0{a_3}^2c_1 \ -2\, f_6{a_2}^3c_1 \ -4\, f_6{a_2}^2a_3c_0 \ -4\, f_7a_0a_2a_3c_3 \ -16\, f_7a_0{a_3}^2c_2 \ -2\, f_7a_1{a_3}^2c_1 \ +f_7{a_2}^3c_2$
$-2\, f_7{a_2}^2a_3c_1 \ -6\, f_7a_2{a_3}^2c_0 + 4\, f_0f_5{a_0}^3a_2 \ -2\, f_0f_5{a_0}^2{a_1}^2 \ -8\, f_0f_7{a_0}^2a_1a_3 \ -2\, f_0f_7{a_0}^2{a_2}^2 \ +4\, f_0f_7a_0{a_1}^2a_2$
$+f_1f_2{a_0}^4 \ +f_1f_3{a_0}^3a_1 \ -f_1f_4{a_0}^3a_2 \ +f_1f_4{a_0}^2{a_1}^2 \ -2\, f_1f_5{a_0}^3a_3 \ -f_1f_5{a_0}^2a_1a_2 \ +f_1f_5a_0{a_1}^3 \ +2\, f_1f_6{a_0}^2a_1a_3$
$-4\, f_1f_6{a_0}^2{a_2}^2 \ +2\, f_1f_6a_0{a_1}^2a_2 \ -7\, f_1f_7{a_0}^2a_2a_3 \ +6\, f_1f_7a_0{a_1}^2a_3 \ -f_1f_7a_0a_1{a_2}^2 \ -f_2f_3{a_0}^3a_2 \ +f_2f_4{a_0}^3a_3$
$-f_2f_4{a_0}^2a_1a_2 \ -5\, f_2f_5{a_0}^2a_1a_3 \ +f_2f_5{a_0}^2{a_2}^2 \ +f_2f_5a_0{a_1}^2a_2 \ +2\, f_2f_6a_0a_1{a_2}^2 \ +4\, f_2f_7{a_0}^2{a_3}^2 \ +2\, f_2f_7a_0a_1a_2a_3$
$-f_2f_7a_0{a_2}^3 \ -{f_3}^2{a_0}^2a_1a_2 \ +f_3f_4{a_0}^2a_1a_3 \ +f_3f_4{a_0}^2{a_2}^2 \ -2\, f_3f_4a_0{a_1}^2a_2 \ +6\, f_3f_5{a_0}^2a_2a_3 \ -7\, f_3f_5a_0{a_1}^2a_3$
$+2\, f_3f_5a_0a_1{a_2}^2 \ -2\, f_3f_6a_0a_1a_2a_3 \ +4\, f_3f_6a_0{a_2}^3 \ -4\, f_3f_7a_0a_1{a_3}^2 \ +5\, f_3f_7a_0{a_2}^2a_3 \ -{f_4}^2{a_0}^2a_2a_3 \ +{f_4}^2a_0{a_1}^2a_3$
$+{f_4}^2a_0a_1{a_2}^2 \ -{f_4}^2{a_1}^3a_2 \ -2\, f_4f_5{a_0}^2{a_3}^2 \ +4\, f_4f_5a_0a_1a_2a_3 \ +2\, f_4f_5a_0{a_2}^3 \ -2\, f_4f_5{a_1}^3a_3 \ -f_4f_5{a_1}^2{a_2}^2$
$+2\, f_4f_6a_0a_1{a_3}^2 \ +2\, f_4f_6a_0{a_2}^2a_3 \ -4\, f_4f_6{a_1}^2a_2a_3 \ +2\, f_4f_6a_1{a_2}^3 \ -2\, f_4f_7a_0a_2{a_3}^2 \ -4\, f_4f_7{a_1}^2{a_3}^2$
$+2\, f_4f_7a_1{a_2}^2a_3 \ +f_4f_7{a_2}^4 \ +14\, {f_5}^2a_0a_1{a_3}^2 \ -4\, {f_5}^2a_0{a_2}^2a_3 \ +{f_5}^2{a_1}^2a_2a_3 \ -{f_5}^2a_1{a_2}^3 \ -24\, f_5f_6a_0a_2{a_3}^2$
$-2\, f_5f_6a_1{a_2}^2a_3 \ +2\, f_5f_6{a_2}^4 \ -50\, f_5f_7a_0{a_3}^3 \ -f_5f_7a_1a_2{a_3}^2 \ +5\, f_5f_7{a_2}^3a_3 \ +4\, {f_6}^2{a_2}^3a_3 \ +12\, f_6f_7{a_2}^2{a_3}^2$
$+8\, {f_7}^2a_2{a_3}^3.$

$\mathbf{eq[21]} = {a_0}^2c_2c_3 \ -a_0a_1{c_2}^2 \ -2\, a_0a_3c_0c_2 - 2\, a_0a_3{c_1}^2 \ +a_1a_2{c_1}^2 \ +a_1a_3c_0c_1 + 2\, f_0{a_0}^3c_1 \ -2\, f_0{a_0}^2a_1c_0 \ +f_1{a_0}^3c_2$
$+f_1{a_0}^2a_1c_1 \ -f_2{a_0}^2a_2c_1 \ +f_2{a_0}^2a_3c_0 \ -2\, f_3{a_0}^2a_1c_3 \ -f_3{a_0}^2a_2c_2 \ +2\, f_3{a_0}^2a_3c_1 \ +2\, f_3a_0{a_1}^2c_2 \ +f_3a_0a_1a_3c_0$
$-f_4{a_0}^2a_2c_3 \ +f_4{a_0}^2a_3c_2 \ +3\, f_4a_0a_1a_3c_1 \ +2\, f_4a_0{a_2}^2c_1 \ -2\, f_4{a_1}^2a_2c_1 \ -f_4{a_1}^2a_3c_0 \ +8\, f_5{a_0}^2a_3c_3 \ -2\, f_5a_0a_1a_2c_3$
$-9\, f_5a_0a_1a_3c_2 \ -10\, f_5a_0{a_3}^2c_0 \ +2\, f_5{a_1}^2a_2c_2 \ -f_5{a_1}^2a_3c_1 \ +f_5a_1a_2a_3c_0 \ +2\, f_6a_0a_2a_3c_2 \ +2\, f_6a_0{a_3}^2c_1$
$-2\, f_6a_1a_2a_3c_1 \ -2\, f_7a_0a_2a_3c_3 \ +7\, f_7a_0{a_3}^2c_2 \ -2\, f_7a_1{a_3}^2c_1 \ +f_7{a_2}^2a_3c_1 \ +f_7a_2{a_3}^2c_0 - 4\, f_0f_5{a_0}^3a_2$
$+2\, f_0f_5{a_0}^2{a_1}^2 \ +8\, f_0f_7{a_0}^2a_1a_3 \ +2\, f_0f_7{a_0}^2{a_2}^2 \ -4\, f_0f_7a_0{a_1}^2a_2 \ -f_1f_2{a_0}^4 \ -f_1f_3{a_0}^3a_1 \ +f_1f_4{a_0}^3a_2$
$-f_1f_4{a_0}^2{a_1}^2 \ +2\, f_1f_5{a_0}^3a_3 \ -f_1f_5{a_0}^2a_1a_2 \ -2\, f_1f_6{a_0}^2a_1a_3 \ +f_1f_7{a_0}^2a_2a_3 \ -2\, f_1f_7a_0{a_1}^2a_3 \ +f_2f_3{a_0}^3a_2$
$-f_2f_4{a_0}^3a_3 \ +f_2f_4{a_0}^2a_1a_2 \ -f_2f_5{a_0}^2a_1a_3 \ +f_2f_5{a_0}^2{a_2}^2 \ -4\, f_2f_7{a_0}^2{a_3}^2 \ +2\, f_2f_7a_0a_1a_2a_3 \ +{f_3}^2{a_0}^2a_1a_2$
$-f_3f_4{a_0}^2a_1a_3 \ -f_3f_4{a_0}^2{a_2}^2 \ +2\, f_3f_4a_0{a_1}^2a_2 \ -6\, f_3f_5{a_0}^2a_2a_3 \ +5\, f_3f_5a_0{a_1}^2a_3 \ -2\, f_3f_6a_0a_1a_2a_3$
$-2\, f_3f_7a_0a_1{a_3}^2 \ +f_3f_7a_0{a_2}^2a_3 \ +{f_4}^2{a_0}^2a_2a_3 \ -{f_4}^2a_0{a_1}^2a_3 \ -{f_4}^2a_0a_1{a_2}^2 \ +{f_4}^2{a_1}^3a_2 \ +2\, f_4f_5{a_0}^2{a_3}^2$
$-4\, f_4f_5a_0a_1a_2a_3 \ -2\, f_4f_5a_0{a_2}^3 \ +f_4f_5{a_1}^3a_3 \ +2\, f_4f_5{a_1}^2{a_2}^2 \ -2\, f_4f_6a_0a_1{a_3}^2 \ -2\, f_4f_6a_0{a_2}^2a_3 \ +2\, f_4f_6{a_1}^2a_2a_3$
$+2\, f_4f_7a_0a_2{a_3}^2 \ +2\, f_4f_7{a_1}^2{a_3}^2 \ -f_4f_7a_1{a_2}^2a_3 \ -14\, {f_5}^2a_0a_1{a_3}^2 \ +2\, {f_5}^2a_0{a_2}^2a_3 \ +5\, {f_5}^2{a_1}^2a_2a_3 \ -{f_5}^2a_1{a_2}^3$

$+12\, f_5 f_6 a_0 a_2 a_3{}^2 - 2\, f_5 f_6 a_1 a_2{}^2 a_3 \;+32\, f_5 f_7 a_0 a_3{}^3 \;-5\, f_5 f_7 a_1 a_2 a_3{}^2 \;- f_5 f_7 a_2{}^3 a_3 \;-2\, f_6 f_7 a_2{}^2 a_3{}^2 \;-2\, f_7{}^2 a_2 a_3{}^3.$

$\mathbf{eq}[\mathbf{22}] = -2\, a_0 a_1 c_2{}^2 \;-2\, a_0 a_2 c_0 c_3 \;-5\, a_0 a_3 c_0 c_2 \;- a_0 a_3 c_1{}^2 \;-2\, a_1{}^2 c_1 c_2 \;+ a_1 a_2 c_0 c_2 \;- a_1 a_2 c_1{}^2 + 2\, f_0 a_0{}^3 c_1$
$-2\, f_0 a_0{}^2 a_1 c_0 \;+ f_1 a_0{}^3 c_2 \;-2\, f_1 a_0{}^2 a_2 c_0 \;+ f_1 a_0 a_1{}^2 c_0 \;+2\, f_2 a_0{}^2 a_1 c_2 \;+ f_2 a_0{}^2 a_3 c_0 \;+ f_2 a_0 a_1 a_2 c_0 \;- f_3 a_0{}^2 a_2 c_2$
$+2\, f_3 a_0{}^2 a_3 c_1 \;+2\, f_3 a_0 a_1{}^2 c_2 \;- f_3 a_0 a_1 a_3 c_0 \;+2\, f_3 a_0 a_2{}^2 c_0 \;-3\, f_4 a_0{}^2 a_2 c_3 \;+ f_4 a_0{}^2 a_3 c_2 \;+2\, f_4 a_0 a_1{}^2 c_3$
$+2\, f_4 a_0 a_1 a_3 c_1 \;+3\, f_4 a_0 a_2{}^2 c_1 \;-2\, f_4 a_1{}^2 a_2 c_1 \;-2\, f_4 a_1{}^2 a_3 c_0 \;+ f_4 a_1 a_2{}^2 c_0 \;+2\, f_5 a_0{}^2 a_3 c_3 \;+4\, f_5 a_0 a_1 a_2 c_3$
$-11\, f_5 a_0 a_1 a_3 c_2 \;+ f_5 a_0 a_2{}^2 c_2 \;-16\, f_5 a_0 a_3{}^2 c_0 \;- f_5 a_1{}^2 a_2 c_2 \;-6\, f_5 a_1{}^2 a_3 c_1 \;+ f_5 a_1 a_2 a_3 c_0 \;+2\, f_6 a_0 a_2{}^2 c_3$
$+8\, f_6 a_0 a_2 a_3 c_2 \;+2\, f_6 a_0 a_3{}^2 c_1 \;+2\, f_6 a_2{}^3 c_1 \;+2\, f_6 a_2{}^2 a_3 c_0 \;+4\, f_7 a_0 a_2 a_3 c_3 \;+16\, f_7 a_0 a_3{}^2 c_2 \;+2\, f_7 a_1 a_3{}^2 c_1 \;- f_7 a_2{}^3 c_2$
$+2\, f_7 a_2{}^2 a_3 c_1 \;+3\, f_7 a_2 a_3{}^2 c_0 - 4\, f_0 f_5 a_0{}^3 a_2 \;+2\, f_0 f_5 a_0{}^2 a_1{}^2 \;+8\, f_0 f_7 a_0{}^2 a_1 a_3 \;+2\, f_0 f_7 a_0{}^2 a_2{}^2 \;-4\, f_0 f_7 a_0 a_1{}^2 a_2$
$- f_1 f_2 a_0{}^4 \;- f_1 f_3 a_0{}^3 a_1 \;+ f_1 f_4 a_0{}^3 a_2 \;- f_1 f_4 a_0{}^2 a_1{}^2 \;+2\, f_1 f_5 a_0{}^3 a_3 \;+ f_1 f_5 a_0{}^2 a_1 a_2 \;- f_1 f_5 a_0 a_1{}^3 \;-2\, f_1 f_6 a_0{}^2 a_1 a_3$
$+4\, f_1 f_6 a_0{}^2 a_2{}^2 \;-2\, f_1 f_6 a_0 a_1{}^2 a_2 \;+7\, f_1 f_7 a_0{}^2 a_2 a_3 \;-6\, f_1 f_7 a_0 a_1{}^2 a_3 \;+ f_1 f_7 a_0 a_1 a_2{}^2 \;+ f_2 f_3 a_0{}^3 a_2 \;- f_2 f_4 a_0{}^3 a_3$
$+ f_2 f_4 a_0{}^2 a_1 a_2 \;+5\, f_2 f_5 a_0{}^2 a_1 a_3 \;- f_2 f_5 a_0{}^2 a_2{}^2 \;- f_2 f_5 a_0 a_1{}^2 a_2 \;-2\, f_2 f_6 a_0 a_1 a_2{}^2 \;-4\, f_2 f_7 a_0{}^2 a_3{}^2 \;-2\, f_2 f_7 a_0 a_1 a_2 a_3$
$+ f_2 f_7 a_0 a_2{}^3 \;+ f_3{}^2 a_0{}^2 a_1 a_2 \;- f_3 f_4 a_0{}^2 a_1 a_3 \;- f_3 f_4 a_0{}^2 a_2{}^2 \;+2\, f_3 f_4 a_0 a_1{}^2 a_2 \;-6\, f_3 f_5 a_0{}^2 a_2 a_3 \;+7\, f_3 f_5 a_0 a_1{}^2 a_3$
$-2\, f_3 f_5 a_0 a_1 a_2{}^2 \;+2\, f_3 f_6 a_0 a_1 a_2 a_3 \;-4\, f_3 f_6 a_0 a_2{}^3 \;+4\, f_3 f_7 a_0 a_1 a_3{}^2 \;-5\, f_3 f_7 a_0 a_2{}^2 a_3 \;+ f_4{}^2 a_0{}^2 a_2 a_3 \;- f_4{}^2 a_0 a_1{}^2 a_3$
$- f_4{}^2 a_0 a_1 a_2{}^2 \;+ f_4{}^2 a_1{}^3 a_2 \;+2\, f_4 f_5 a_0{}^2 a_3{}^2 \;-4\, f_4 f_5 a_0 a_1 a_2 a_3 \;-2\, f_4 f_5 a_0 a_2{}^3 \;+2\, f_4 f_5 a_1{}^3 a_3 \;+ f_4 f_5 a_1{}^2 a_2{}^2$
$-2\, f_4 f_6 a_0 a_1 a_3{}^2 \;-2\, f_4 f_6 a_0 a_2{}^2 a_3 \;+4\, f_4 f_6 a_1{}^2 a_2 a_3 \;-2\, f_4 f_6 a_1 a_2{}^3 \;+2\, f_4 f_7 a_0 a_2 a_3{}^2 \;+4\, f_4 f_7 a_1{}^2 a_3{}^2$
$-2\, f_4 f_7 a_1 a_2{}^2 a_3 \;- f_4 f_7 a_2{}^4 \;-14\, f_5{}^2 a_0 a_1 a_3{}^2 \;+4\, f_5{}^2 a_0 a_2{}^2 a_3 \;- f_5{}^2 a_1{}^2 a_2 a_3 \;+ f_5{}^2 a_1 a_2{}^3 \;+24\, f_5 f_6 a_0 a_2 a_3{}^2$
$+2\, f_5 f_6 a_1 a_2{}^2 a_3 \;-2\, f_5 f_6 a_2{}^4 \;+50\, f_5 f_7 a_0 a_3{}^3 \;+ f_5 f_7 a_1 a_2 a_3{}^2 \;-5\, f_5 f_7 a_2{}^3 a_3 \;-4\, f_6{}^2 a_2{}^3 a_3 \;-12\, f_6 f_7 a_2{}^2 a_3{}^2$
$-8\, f_7{}^2 a_2 a_3{}^3.$

$\mathbf{eq}[\mathbf{23}] = a_0 a_2 c_1 c_2 \;- a_0 a_3 c_0 c_2 \;-2\, a_0 a_3 c_1{}^2 \;+ a_1 a_2 c_1{}^2 \;+ a_1 a_3 c_0 c_1 + 2\, f_0 a_0{}^3 c_1 \;-2\, f_0 a_0{}^2 a_1 c_0 \;+ f_1 a_0{}^3 c_2$
$+ f_1 a_0{}^2 a_1 c_1 \;- f_2 a_0{}^2 a_2 c_1 \;+ f_2 a_0{}^2 a_3 c_0 \;-2\, f_3 a_0{}^2 a_1 c_3 \;- f_3 a_0{}^2 a_2 c_2 \;+2\, f_3 a_0{}^2 a_3 c_1 \;+2\, f_3 a_0 a_1{}^2 c_2 \;+ f_3 a_0 a_1 a_3 c_0$
$- f_4 a_0{}^2 a_2 c_3 \;+ f_4 a_0{}^2 a_3 c_2 \;+3\, f_4 a_0 a_1 a_3 c_1 \;+2\, f_4 a_0 a_2{}^2 c_1 \;-2\, f_4 a_1{}^2 a_2 c_1 \;- f_4 a_1{}^2 a_3 c_0 \;+8\, f_5 a_0{}^2 a_3 c_3 \;-2\, f_5 a_0 a_1 a_2 c_3$
$-7\, f_5 a_0 a_1 a_3 c_2 \;- f_5 a_0 a_2{}^2 c_2 \;-10\, f_5 a_0 a_3{}^2 c_0 \;+2\, f_5 a_1{}^2 a_2 c_2 \;- f_5 a_1{}^2 a_3 c_1 + f_5 a_1 a_2 a_3 c_0 \;+2\, f_6 a_0 a_3{}^2 c_1$
$-2\, f_6 a_1 a_2 a_3 c_1 \;-2\, f_7 a_0 a_2 a_3 c_3 \;+4\, f_7 a_0 a_3{}^2 c_2 \;-2\, f_7 a_1 a_3{}^2 c_1 \;+ f_7 a_2{}^2 a_3 c_1 \;+ f_7 a_2 a_3{}^2 c_0 - 4\, f_0 f_5 a_0{}^3 a_2$
$+2\, f_0 f_5 a_0{}^2 a_1{}^2 \;+8\, f_0 f_7 a_0{}^2 a_1 a_3 \;+2\, f_0 f_7 a_0{}^2 a_2{}^2 \;-4\, f_0 f_7 a_0 a_1{}^2 a_2 \;- f_1 f_2 a_0{}^4 \;- f_1 f_3 a_0{}^3 a_1 \;+ f_1 f_4 a_0{}^3 a_2$
$- f_1 f_4 a_0{}^2 a_1{}^2 \;+2\, f_1 f_5 a_0{}^3 a_3 \;- f_1 f_5 a_0{}^2 a_1 a_2 \;-2\, f_1 f_6 a_0{}^2 a_1 a_3 \;+ f_1 f_7 a_0{}^2 a_2 a_3 \;-2\, f_1 f_7 a_0 a_1{}^2 a_3 \;+ f_2 f_3 a_0{}^3 a_2$
$- f_2 f_4 a_0{}^3 a_3 \;+ f_2 f_4 a_0{}^2 a_1 a_2 \;- f_2 f_5 a_0{}^2 a_1 a_3 \;+ f_2 f_5 a_0{}^2 a_2{}^2 \;-4\, f_2 f_7 a_0{}^2 a_3{}^2 \;+2\, f_2 f_7 a_0 a_1 a_2 a_3 \;+ f_3{}^2 a_0{}^2 a_1 a_2$
$- f_3 f_4 a_0{}^2 a_1 a_3 \;- f_3 f_4 a_0{}^2 a_2{}^2 \;+2\, f_3 f_4 a_0 a_1{}^2 a_2 \;-6\, f_3 f_5 a_0{}^2 a_2 a_3 \;+5\, f_3 f_5 a_0 a_1{}^2 a_3 \;-2\, f_3 f_6 a_0 a_1 a_2 a_3$
$-2\, f_3 f_7 a_0 a_1 a_3{}^2 \;+ f_3 f_7 a_0 a_2{}^2 a_3 \;+ f_4{}^2 a_0{}^2 a_2 a_3 \;- f_4{}^2 a_0 a_1{}^2 a_3 \;- f_4{}^2 a_0 a_1 a_2{}^2 \;+ f_4{}^2 a_1{}^3 a_2 \;+2\, f_4 f_5 a_0{}^2 a_3{}^2$
$-4\, f_4 f_5 a_0 a_1 a_2 a_3 \;-2\, f_4 f_5 a_0 a_2{}^3 \;+ f_4 f_5 a_1{}^3 a_3 \;+2\, f_4 f_5 a_1{}^2 a_2{}^2 \;-2\, f_4 f_6 a_0 a_1 a_3{}^2 \;-2\, f_4 f_6 a_0 a_2{}^2 a_3 \;+2\, f_4 f_6 a_1{}^2 a_2 a_3$
$+2\, f_4 f_7 a_0 a_2 a_3{}^2 \;+2\, f_4 f_7 a_1{}^2 a_3{}^2 \;- f_4 f_7 a_1 a_2{}^2 a_3 \;-14\, f_5{}^2 a_0 a_1 a_3{}^2 \;+2\, f_5{}^2 a_0 a_2{}^2 a_3 \;+5\, f_5{}^2 a_1{}^2 a_2 a_3 \;- f_5{}^2 a_1 a_2{}^3$
$+12\, f_5 f_6 a_0 a_2 a_3{}^2 - 2\, f_5 f_6 a_1 a_2{}^2 a_3 \;+32\, f_5 f_7 a_0 a_3{}^3 \;-5\, f_5 f_7 a_1 a_2 a_3{}^2 \;- f_5 f_7 a_2{}^3 a_3 \;-2\, f_6 f_7 a_2{}^2 a_3{}^2 \;-2\, f_7{}^2 a_2 a_3{}^3.$

$\mathbf{eq}[\mathbf{24}] = a_0{}^2 c_1 c_3 \;+ a_0 a_1 c_0 c_3 \;-3\, a_0 a_1 c_1 c_2 \;+2\, a_0 a_2 c_0 c_2 \;-10\, a_0 a_3 c_0 c_1 \;- a_1{}^2 c_0 c_2 \;-2\, a_1{}^2 c_1{}^2 + 8\, f_0 a_0{}^3 c_0$
$+ f_1 a_0{}^3 c_1 \;- f_1 a_0{}^2 a_1 c_0 \;+2\, f_2 a_0{}^3 c_2 \;+4\, f_2 a_0{}^2 a_1 c_1 \;-2\, f_2 a_0{}^2 a_2 c_0 \;+2\, f_3 a_0{}^2 a_1 c_2 \;- f_3 a_0{}^2 a_2 c_1 \;+8\, f_3 a_0{}^2 a_3 c_0$
$+4\, f_3 a_0 a_1{}^2 c_1 \;- f_3 a_0 a_1 a_2 c_0 \;-5\, f_4 a_0{}^2 a_1 c_3 \;-2\, f_4 a_0{}^2 a_2 c_2 \;+ f_4 a_0{}^2 a_3 c_1 \;+7\, f_4 a_0 a_1{}^2 c_2 \;+12\, f_4 a_0 a_1 a_3 c_0$
$+2\, f_4 a_0 a_2{}^2 c_0 \;+4\, f_4 a_1{}^3 c_1 \;- f_4 a_1{}^2 a_2 c_0 \;+14\, f_5 a_0{}^2 a_2 c_3 \;+2\, f_5 a_0{}^2 a_3 c_2 \;-14\, f_5 a_0 a_1 a_2 c_2 \;-3\, f_5 a_0 a_1 a_3 c_1$
$-16\, f_5 a_0 a_2{}^2 c_1 \;+3\, f_5 a_1{}^2 a_2 c_1 \;-4\, f_5 a_1{}^2 a_3 c_0 \;+18\, f_6 a_0{}^2 a_3 c_3 \;-2\, f_6 a_0 a_1 a_2 c_3 \;-14\, f_6 a_0 a_1 a_3 c_2 \;-2\, f_6 a_0 a_2{}^2 c_2$
$-10\, f_6 a_0 a_3{}^2 c_0 \;+2\, f_6 a_1{}^2 a_2 c_2 \;+2\, f_7 a_0 a_2{}^2 c_3 \;-10\, f_7 a_0 a_2 a_3 c_2 \;+31\, f_7 a_0 a_3{}^2 c_1 \;+4\, f_7 a_1{}^2 a_3 c_2 \;-2\, f_7 a_1 a_2 a_3 c_1$
$+3\, f_7 a_1 a_3{}^2 c_0 \;- f_7 a_2{}^3 c_1 \;- f_7 a_2{}^2 a_3 c_0 - 16\, f_0 f_6 a_0{}^3 a_2 \;+8\, f_0 f_6 a_0{}^2 a_1{}^2 \;-24\, f_0 f_7 a_0{}^3 a_3 \;-8\, f_0 f_7 a_0{}^2 a_1 a_2$
$+8\, f_0 f_7 a_0 a_1{}^3 \;-2\, f_1 f_5 a_0{}^3 a_2 \;+ f_1 f_5 a_0{}^2 a_1{}^2 \;+4\, f_1 f_7 a_0{}^2 a_1 a_3 \;+ f_1 f_7 a_0{}^2 a_2{}^2 \;-2\, f_1 f_7 a_0 a_1{}^2 a_2 \;-2\, f_2{}^2 a_0{}^4$
$-4\, f_2 f_3 a_0{}^3 a_1 \;+4\, f_2 f_4 a_0{}^3 a_2 - 4\, f_2 f_4 a_0{}^2 a_1{}^2 \;+2\, f_2 f_5 a_0{}^3 a_3 \;-2\, f_2 f_5 a_0{}^2 a_1 a_2 \;-8\, f_2 f_6 a_0{}^2 a_1 a_3 \;+2\, f_2 f_6 a_0{}^2 a_2{}^2$
$+12\, f_2 f_7 a_0{}^2 a_2 a_3 \;-8\, f_2 f_7 a_0 a_1{}^2 a_3 \;-2\, f_2 f_7 a_0 a_1 a_2{}^2 \;-2\, f_3{}^2 a_0{}^2 a_1{}^2 \;+4\, f_3 f_4 a_0{}^2 a_1 a_2 \;-4\, f_3 f_4 a_0 a_1{}^3$
$+2\, f_3 f_5 a_0{}^2 a_1 a_3 \;+2\, f_3 f_5 a_0{}^2 a_2{}^2 \;-3\, f_3 f_5 a_0 a_1{}^2 a_2 \;-16\, f_3 f_6 a_0{}^2 a_2 a_3 \;+2\, f_3 f_6 a_0 a_1 a_2{}^2 \;-24\, f_3 f_7 a_0{}^2 a_3{}^2$
$- f_3 f_7 a_0 a_2{}^3 - 2\, f_4{}^2 a_0{}^2 a_2{}^2 \;+4\, f_4{}^2 a_0 a_1{}^2 a_2 - 2\, f_4{}^2 a_1{}^4 \;-4\, f_4 f_5 a_0{}^2 a_2 a_3 \;+13\, f_4 f_5 a_0 a_1{}^2 a_3 \;- f_4 f_5 a_0 a_1 a_2{}^2$
$-3\, f_4 f_5 a_1{}^3 a_2 \;-18\, f_4 f_6 a_0 a_1 a_2 a_3 \;-2\, f_4 f_6 a_0 a_2{}^3 \;+2\, f_4 f_6 a_1{}^2 a_2{}^2 \;-35\, f_4 f_7 a_0 a_1 a_3{}^2 \;-11\, f_4 f_7 a_0 a_2{}^2 a_3$
$+6\, f_4 f_7 a_1{}^2 a_2 a_3 \;+ f_4 f_7 a_1 a_2{}^3 \;+4\, f_5{}^2 a_0{}^2 a_3{}^2 \;-32\, f_5{}^2 a_0 a_1 a_2 a_3 \;+16\, f_5{}^2 a_0 a_2{}^3 \;+ f_5{}^2 a_1{}^3 a_3 \;- f_5{}^2 a_1{}^2 a_2{}^2$
$-32\, f_5 f_6 a_0 a_1 a_3{}^2 \;+26\, f_5 f_6 a_0 a_2{}^2 a_3 \;+8\, f_5 f_6 a_1{}^2 a_2 a_3 \;-8\, f_5 f_7 a_0 a_2 a_3{}^2 \;+12\, f_5 f_7 a_1{}^2 a_3{}^2 \;+3\, f_5 f_7 a_1 a_2{}^2 a_3$
$+ f_5 f_7 a_2{}^4 \;+20\, f_6{}^2 a_0 a_2 a_3{}^2 \;+30\, f_6 f_7 a_0 a_3{}^3 \;-6\, f_6 f_7 a_1 a_2 a_3{}^2 \;+2\, f_6 f_7 a_2{}^3 a_3 \;-8\, f_7{}^2 a_1 a_3{}^3.$

$\mathbf{eq[25]} = a_0{}^2c_2{}^2 +2\,a_0a_1c_1c_2 +4\,a_0a_3c_0c_1 +a_1{}^2c_1{}^2 - 4\,f_0a_0{}^3c_0 -2\,f_2a_0{}^3c_2 -2\,f_2a_0{}^2a_1c_1 -2\,f_3a_0{}^2a_1c_2$
$-4\,f_3a_0{}^2a_3c_0 -2\,f_3a_0a_1{}^2c_1 +2\,f_4a_0{}^2a_1c_3 +2\,f_4a_0{}^2a_2c_2 -4\,f_4a_0a_1{}^2c_2 -6\,f_4a_0a_1a_3c_0 -2\,f_4a_1{}^3c_1$
$-4\,f_5a_0{}^2a_2c_3 +4\,f_5a_0{}^2a_3c_2 +2\,f_5a_0a_1a_2c_2 +4\,f_5a_0a_1a_3c_1 +4\,f_5a_0a_2{}^2c_1 -2\,f_5a_1{}^2a_2c_1 -8\,f_6a_0{}^2a_3c_3$
$+4\,f_6a_0a_1a_3c_2 +4\,f_6a_0a_3{}^2c_0 +2\,f_7a_0a_2a_3c_2 -12\,f_7a_0a_3{}^2c_1 -4\,f_7a_1{}^2a_3c_2 -2\,f_7a_1a_2a_3c_1$
$-4\,f_7a_1a_3{}^2c_0 + 8\,f_0f_6a_0{}^3a_2 -4\,f_0f_6a_0{}^2a_1{}^2 +12\,f_0f_7a_0{}^3a_3 +4\,f_0f_7a_0{}^2a_1a_2 -4\,f_0f_7a_0a_1{}^3 +f_2{}^2a_0{}^4$
$+2\,f_2f_3a_0{}^3a_1 -2\,f_2f_4a_0{}^3a_2 +2\,f_2f_4a_0{}^2a_1{}^2 -4\,f_2f_5a_0{}^3a_3 +2\,f_2f_5a_0{}^2a_1a_2 +4\,f_2f_6a_0{}^2a_1a_3 -2\,f_2f_7a_0{}^2a_2a_3$
$+4\,f_2f_7a_0a_1{}^2a_3 +f_3{}^2a_0{}^2a_1{}^2 -2\,f_3f_4a_0{}^2a_1a_2 +2\,f_3f_4a_0a_1{}^3 -4\,f_3f_5a_0{}^2a_1a_3 +2\,f_3f_5a_0a_1{}^2a_2 +8\,f_3f_6a_0{}^2a_2a_3$
$+12\,f_3f_7a_0{}^2a_3{}^2 +2\,f_3f_7a_0a_1a_2a_3 +f_4{}^2a_0{}^2a_2{}^2 - 2\,f_4{}^2a_0a_1{}^2a_2 +f_4{}^2a_1{}^4 +4\,f_4f_5a_0{}^2a_2a_3 -8\,f_4f_5a_0a_1{}^2a_3$
$+2\,f_4f_5a_1{}^3a_2 +8\,f_4f_6a_0a_1a_2a_3 +18\,f_4f_7a_0a_1a_3{}^2 +2\,f_4f_7a_0a_2{}^2a_3 -2\,f_4f_7a_1{}^2a_2a_3 +4\,f_5{}^2a_0{}^2a_3{}^2$
$+4\,f_5{}^2a_0a_1a_2a_3 -4\,f_5{}^2a_0a_2{}^3 +f_5{}^2a_1{}^2a_2{}^2 +8\,f_5f_6a_0a_1a_3{}^2 -8\,f_5f_6a_0a_2{}^2a_3 +4\,f_5f_7a_0a_2a_3{}^2 -8\,f_5f_7a_1{}^2a_3{}^2$
$+2\,f_5f_7a_1a_2{}^2a_3 -8\,f_6{}^2a_0a_2a_3{}^2 -12\,f_6f_7a_0a_3{}^3 +8\,f_6f_7a_1a_2a_3{}^2 +12\,f_7{}^2a_1a_3{}^3 +f_7{}^2a_2{}^2a_3{}^2.$

$\mathbf{eq[26]} = a_0{}^2c_0c_3 -a_0{}^2c_1c_2 -a_0a_1c_1{}^2 -3\,a_0a_2c_0c_1 +a_0a_3c_0{}^2 +a_1{}^2c_0c_1 - 2\,f_1a_0{}^3c_0 +f_2a_0{}^3c_1 -f_2a_0{}^2a_1c_0$
$+f_3a_0{}^2a_1c_1 +2\,f_3a_0{}^2a_2c_0 -f_3a_0a_1{}^2c_0 -f_4a_0{}^2a_2c_1 -2\,f_4a_0{}^2a_3c_0 +f_4a_0a_1{}^2c_1 +3\,f_4a_0a_1a_2c_0 -f_4a_1{}^3c_0$
$+3\,f_5a_0{}^2a_1c_3 +2\,f_5a_0{}^2a_2c_2 -2\,f_5a_0{}^2a_3c_1 -4\,f_5a_0a_1{}^2c_2 -5\,f_5a_0a_1a_3c_0 +3\,f_5a_0a_2{}^2c_0 -f_5a_1{}^3c_1 -f_5a_1{}^2a_2c_0$
$-2\,f_6a_0{}^2a_2c_3 +2\,f_6a_0a_1a_2c_2 +2\,f_6a_0a_1a_3c_1 +6\,f_6a_0a_2{}^2c_1 -2\,f_6a_1{}^2a_2c_1 +5\,f_7a_0{}^2a_3c_3 +f_7a_0a_1a_2c_3$
$-9\,f_7a_0a_1a_3c_2 -f_7a_0a_2{}^2c_2 -16\,f_7a_0a_3{}^2c_0 +f_7a_1{}^2a_2c_2 -2\,f_7a_1{}^2a_3c_1 +2\,f_7a_1a_2a_3c_0 + 4\,f_1f_6a_0{}^3a_2$
$-2\,f_1f_6a_0{}^2a_1{}^2 +6\,f_1f_7a_0{}^3a_3 +2\,f_1f_7a_0{}^2a_1a_2 -2\,f_1f_7a_0a_1{}^3 -2\,f_2f_5a_0{}^3a_2 +f_2f_5a_0{}^2a_1{}^2 +4\,f_2f_7a_0{}^2a_1a_3$
$+f_2f_7a_0{}^2a_2{}^2 -2\,f_2f_7a_0a_1{}^2a_2 -2\,f_3f_5a_0{}^2a_1a_2 +f_3f_5a_0a_1{}^3 -4\,f_3f_6a_0{}^2a_2{}^2 +2\,f_3f_6a_0a_1{}^2a_2 -6\,f_3f_7a_0{}^2a_2a_3$
$+4\,f_3f_7a_0a_1{}^2a_3 -f_3f_7a_0a_1a_2{}^2 +2\,f_4f_5a_0{}^2a_2{}^2 -3\,f_4f_5a_0a_1{}^2a_2 +f_4f_5a_1{}^4 +4\,f_4f_6a_0{}^2a_2a_3 -2\,f_4f_6a_0a_1{}^2a_3$
$-4\,f_4f_6a_0a_1a_2{}^2 +2\,f_4f_6a_1{}^3a_2 +6\,f_4f_7a_0{}^2a_3{}^2 -8\,f_4f_7a_0a_1a_2a_3 -f_4f_7a_0a_2{}^3 +2\,f_4f_7a_1{}^3a_3 +f_4f_7a_1{}^2a_2{}^2$
$+4\,f_5{}^2a_0{}^2a_2a_3 -8\,f_5{}^2a_0a_1{}^2a_3 +f_5{}^2a_0a_1a_2{}^2 +f_5{}^2a_1{}^3a_2 +10\,f_5f_6a_0a_1a_2a_3 -6\,f_5f_6a_0a_2{}^3 +2\,f_5f_6a_1{}^2a_2{}^2$
$-3\,f_5f_7a_0a_1a_3{}^2 -f_5f_7a_0a_2{}^2a_3 +3\,f_5f_7a_1{}^2a_2a_3 -4\,f_6{}^2a_0a_2{}^2a_3 +20\,f_6f_7a_0a_2a_3{}^2 -4\,f_6f_7a_1a_2{}^2a_3$
$+39\,f_7{}^2a_0a_3{}^3 -5\,f_7{}^2a_1a_2a_3{}^2 -f_7{}^2a_2{}^3a_3.$

$\mathbf{eq[27]} = a_0{}^2c_1c_3 -3\,a_0a_1c_1c_2 +2\,a_0a_2c_0c_2 -10\,a_0a_3c_0c_1 -2\,a_1{}^2c_1{}^2 +a_1a_2c_0c_1 +a_1a_3c_0{}^2 + 8\,f_0a_0{}^3c_0$
$+f_1a_0{}^3c_1 -f_1a_0{}^2a_1c_0 +2\,f_2a_0{}^3c_2 +4\,f_2a_0{}^2a_1c_1 -2\,f_2a_0{}^2a_2c_0 +2\,f_3a_0{}^2a_1c_2 -f_3a_0{}^2a_2c_1 +8\,f_3a_0{}^2a_3c_0$
$+4\,f_3a_0a_1{}^2c_1 -f_3a_0a_1a_2c_0 -5\,f_4a_0{}^2a_1c_3 -2\,f_4a_0{}^2a_2c_2 +f_4a_0{}^2a_3c_1 +7\,f_4a_0a_1{}^2c_2 +12\,f_4a_0a_1a_3c_0$
$+2\,f_4a_0a_2{}^2c_0 +4\,f_4a_1{}^3c_1 -f_4a_1{}^2a_2c_0 +14\,f_5a_0{}^2a_2c_3 +2\,f_5a_0{}^2a_3c_2 -14\,f_5a_0a_1a_2c_2 -3\,f_5a_0a_1a_3c_1$
$-16\,f_5a_0a_2{}^2c_1 +3\,f_5a_1{}^2a_2c_1 -2\,f_5a_1{}^2a_3c_0 -f_5a_1a_2{}^2c_0 +18\,f_6a_0{}^2a_3c_3 -2\,f_6a_0a_1a_2c_3 -14\,f_6a_0a_1a_3c_2$
$-2\,f_6a_0a_2{}^2c_2 -10\,f_6a_0a_3{}^2c_0 +2\,f_6a_1{}^2a_2c_2 -2\,f_6a_1a_2a_3c_0 +2\,f_7a_0a_2{}^2c_3 -10\,f_7a_0a_2a_3c_2 +31\,f_7a_0a_3{}^2c_1$
$+4\,f_7a_1{}^2a_3c_2 -2\,f_7a_1a_2a_3c_1 -f_7a_2{}^3c_1 -f_7a_2{}^2a_3c_0 - 16\,f_0f_6a_0{}^3a_2 +8\,f_0f_6a_0{}^2a_1{}^2 -24\,f_0f_7a_0{}^3a_3$
$-8\,f_0f_7a_0{}^2a_1a_2 +8\,f_0f_7a_0a_1{}^3 -2\,f_1f_5a_0{}^3a_2 +f_1f_5a_0{}^2a_1{}^2 +4\,f_1f_7a_0{}^2a_1a_3 +f_1f_7a_0{}^2a_2{}^2 -2\,f_1f_7a_0a_1{}^2a_2$
$-2\,f_2{}^2a_0{}^4 -4\,f_2f_3a_0{}^3a_1 +4\,f_2f_4a_0{}^3a_2 - 4\,f_2f_4a_0{}^2a_1{}^2 +2\,f_2f_5a_0{}^3a_3 -2\,f_2f_5a_0{}^2a_1a_2 -8\,f_2f_6a_0{}^2a_1a_3$
$+2\,f_2f_6a_0{}^2a_2{}^2 +12\,f_2f_7a_0{}^2a_2a_3 -8\,f_2f_7a_0a_1{}^2a_3 -2\,f_2f_7a_0a_1a_2{}^2 -2\,f_3{}^2a_0{}^2a_1{}^2 +4\,f_3f_4a_0{}^2a_1a_2 -4\,f_3f_4a_0a_1{}^3$
$+2\,f_3f_5a_0{}^2a_1a_3 +2\,f_3f_5a_0{}^2a_2{}^2 -3\,f_3f_5a_0a_1{}^2a_2 -16\,f_3f_6a_0{}^2a_2a_3 +2\,f_3f_6a_0a_1a_2{}^2 -24\,f_3f_7a_0{}^2a_3{}^2$
$-f_3f_7a_0a_2{}^3 - 2\,f_4{}^2a_0{}^2a_2{}^2 +4\,f_4{}^2a_0a_1{}^2a_2 - 2\,f_4{}^2a_1{}^4 -4\,f_4f_5a_0{}^2a_2a_3 +13\,f_4f_5a_0a_1{}^2a_3 -f_4f_5a_0a_1a_2{}^2$
$-3\,f_4f_5a_1{}^3a_2 -18\,f_4f_6a_0a_1a_2a_3 -2\,f_4f_6a_0a_2{}^3 +2\,f_4f_6a_1{}^2a_2{}^2 -35\,f_4f_7a_0a_1a_3{}^2 -11\,f_4f_7a_0a_2{}^2a_3$
$+6\,f_4f_7a_1{}^2a_2a_3 +f_4f_7a_1a_2{}^3 +4\,f_5{}^2a_0{}^2a_3{}^2 -32\,f_5{}^2a_0a_1a_2a_3 +16\,f_5{}^2a_0a_2{}^3 +f_5{}^2a_1{}^3a_3 -f_5{}^2a_1{}^2a_2{}^2$
$-32\,f_5f_6a_0a_1a_3{}^2 +26\,f_5f_6a_0a_2{}^2a_3 +8\,f_5f_6a_1{}^2a_2a_3 -8\,f_5f_7a_0a_2a_3{}^2 +12\,f_5f_7a_1{}^2a_3{}^2 +3\,f_5f_7a_1a_2{}^2a_3$
$+f_5f_7a_2{}^4 +20\,f_6{}^2a_0a_2a_3{}^2 +30\,f_6f_7a_0a_3{}^3 -6\,f_6f_7a_1a_2a_3{}^2 +2\,f_6f_7a_2{}^3a_3 -8\,f_7{}^2a_1a_3{}^3.$

Subject to Conjecture 2.4, and by analogy with the defining equation of the Kummer surface in genus 2, which is only quadratic in k_3, we make the following conjecture.

Conjecture 5.3 *The equations of Theorem 5.2 are a set of defining equations for the Kummer variety in genus 3.*

It would be nice to be able to check all quartic monomials, including those quartic in c_0, c_1, c_2, c_3. However, these are beyond storage capacity. Note that even to verify that the above are identities cannot be done all at once; one need to slice each of the above into portions and use intermediary simplification steps.

Note that one consequence of equation one, $eq[1] = 0$, above, is that there is a unique linear relation between the quadratic monomials in the $\{\kappa_i\} = \{a_i\} \cup \{c_i\}$ of Theorem 3.13. This has a surprising consequence for the even functions of $L(4\Theta)$, which does not happen in genus 1 or 2.

Theorem 5.4 *The* $(4^g + 2^g)/2 = 36$ *quadratic monomials in the* κ_i *of Theorem 3.13 span a* $((4^g + 2^g)/2 - 1) = 35$*-dimensional subspace of the* $(4^g + 2^g)/2 = 36$*-dimensional space of even functions of* $L(4\Theta)$.

Proof Our method of finding all of the equations in Theorem 5.2 included finding all possible quadratic relations on $L(2\Theta)$, of which there is precisely one. There are, in total, $(4^g + 2^g)/2$ quadratic monomials in the κ_i of Theorem 3.13, all of which are even functions in $L(4\Theta)$, and which span one dimension less, due to the above linear dependency. We know that the set of even functions of $L(4\Theta)$ has dimension $(4^g + 2^g)/2$ by Lemma 2.6. $\square$

Contrast this with genus 1 and 2. In any genus, there are $(4^g+2^g)/2$ quadratic monomials in the coordinate functions of $L(2\Theta)$, and the space of even functions of $L(4\Theta)$ is also $(4^g + 2^g)/2$. In genus 1 and 2, the quadratic monomials in the coordinate functions of $L(2\Theta)$ are linearly independent, and so give a basis for the even functions of $L(4\Theta)$. For example, in genus 1, the $\kappa_0 = 1,\ \kappa_1 = x$ of (2.3) give the three quadratic monomials $\kappa_0^2, \kappa_0\kappa_1, \kappa_1^2$, which give a basis for the span of the three even functions z_0, z_1, z_3 of (2.3). Similarly in genus 2 the $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ of (2.8) give the 10 quadratic monomials $\kappa_0^2, \kappa_0\kappa_1, \kappa_0\kappa_2, \kappa_0\kappa_3, \kappa_1^2, \kappa_1\kappa_2, \kappa_1\kappa_3, \kappa_2^2, \kappa_2\kappa_3, \kappa_3^2$ which span the same space as the 10 even function of $L(4\Theta)$ given by z_0, z_3, z_4, z_5, $z_{10}, z_{11}, z_{12}, z_{13}, z_{14}, z_{15}$ of (2.10). Hence, genus 3 is the first genus for which there is still an even function to be found in $L(4\Theta)$ beyond the quadratic monomials of the basis for $L(2\Theta)$.

5.3 Relations in $L(4\Theta)$ for Genus 3.

Since we have not given a complete basis for $L(3\Theta)$ or $L(4\Theta)$ we cannot of course give a complete set of defining equations in either case. Theorem 3.14 describes what portion of $L(3\Theta)$ we know. For $L(4\Theta)$, we have the 35 even functions e_{ij} of Theorem 5.4, given by the monomials

$$e_{ij} = \kappa_i \kappa_j, \quad 0 \leq i \leq j \leq 7, \ (i,j) \neq (0,7). \tag{5.5}$$

We have excluded $(i,j) = (0,7)$ since $\kappa_0\kappa_7$, which is the same as $a_0 c_3$, is linearly dependent on the other e_{ij} by Theorem 5.2. There are also the b_0, b_1, b_2 of (3.11), giving a total of 38 independent members of the $4^g = 64$-dimensional space $L(4\Theta)$. Twelve of the missing members of the basis for $L(4\Theta)$ are the missing members of the basis for $L(3\Theta)$. Recall from Lemma 2.3 that either $L(3\Theta)$ or $L(4\Theta)$ yields an embedding of the Jacobian variety, with the $L(3\Theta)$ embedding in $\mathbf{P}^{26}$ having defining equations at most cubic, and the $L(4\Theta)$ embedding in $\mathbf{P}^{63}$ having quadratic defining equations, as a special case of a theorem of Mumford ([26], I), which states that the Jacobian can be expressed as an intersection of quadratics.

Many of these are easy to find. First of all, there are the curve-independent equations:

$$e_{ij} e_{k\ell} = e_{ik} e_{j\ell}, \quad i,j,k,\ell \in \{0, \ldots, 7\}, \ j \neq k. \tag{5.6}$$

These are analogous to the equation $z_1^2 = z_0 z_3$ in (2.5) for the $\mathbf{P}^3$ embedding $z_0, \ldots, z_3$ of an elliptic curve given in (2.3), and to equations of the form $z_0 z_{14} = z_5^2$ for the $\mathbf{P}^{15}$ embedding of the Jacobian of a curve of genus 2 described in (2.10). In addition, every one of the quartic equations in Theorem 5.2 induces a quadratic form on the e_{ij} obtained by replacing quadratic monomials $\kappa_i\kappa_j$ by e_{ij}, and by further using $\mathbf{eq}[\mathbf{1}]$ to eliminate any occurrence of $\kappa_0\kappa_7$.

Chapter 6

Future Applications

Much of this section will describe what should be true, with outlines of a strategy rather than proofs.

An explicit embedding of the Jacobian has potential applications to Chabauty's Theorem ([9] and Theorem 13.1.2 in Cassels and Flynn [8]), which states that the number of rational points on a curve over a number field is finite if the rank of the Jacobian is less than the genus of the curve. One approach to implementing Chabauty's Theorem in practice has been to derive explicit local power series expansions from the formal group. In genus 2, it has proved useful to have an explicit embedding of the Jacobian variety, from which the terms of the formal group can be derived, and there are now many genus 2 applications of Chabauty's Theorem in the literature ([4], [9], [10], [13], [15], [16], [17], [19], and [29]).

We shall concentrate on the potential of our results eventually to be applied to give the explicit computation of height constants in genus 3. Much work has already been done in this area for genus 1 and genus 2, which gives rise to applications involving Selmer groups and descent techniques ([12], [14], [30], [31], [33], [34], and [35]).

We first briefly summarise how these height constants can be computed in genus 1 and 2. We first recall the standard definition of a height function on an Abelian group.

Definition 6.1 *Let G be an abelian group. A* height function *H on G is a map $H : G \mapsto \mathbb{R}^+$ satisfying:*

(1). There exists a constant C_1 such that, for all $P, Q \in G$,

$$H(P+Q)H(P-Q) \leqslant C_1 H(P)^2 H(Q)^2.$$

(2). There exists a constant C_2 such that, for all $P \in G$, $H(2P) \geqslant H(P)^4/C_2$.

(3). For any constant C_3, the set $\{P \in G : H(P) \leqslant C_3\}$ is finite.

The constants C_1, C_2 depend only on the group G and the height function H, and we shall refer to them as the height constants.

The following property of abelian groups with a height function is proved in [32], p.199.

Lemma 6.2 *Let G be an abelian group with height function H, such that $G/2G$ is a finite set: $\{Q_1, \ldots, Q_n\}$, say. Then G is finitely generated. Explicitly, if $\epsilon = min\{H(P) : P \in G\}$, and $C_1' = max\{H(Q_i)^2 : 1 \leqslant i \leqslant n\} \cdot C_1/\epsilon$, then G is generated by the finite set: $\{P \in G : H(P) \leqslant \sqrt{C_1' C_2}\} \cup \{Q_1, \ldots, Q_n\}$.*

When K is a number field, there is a standard function H which may be defined on $\mathbf{P}^n(K)$.

Definition 6.3 *Let $\mathbf{x} = (x_i) \in \mathbf{P}^n(K)$. Define $H(\mathbf{x}) = \left(\prod_v \max_{0 \leqslant i \leqslant n} |x_i|_v\right)^{1/[K:\mathbb{Q}]}$, where $\prod_v$ is over all valuations on K which extend the usual valuations $|\ |_p$ and $|\ |_\infty$ on $\mathbb{Q}$.*

When $K = \mathbb{Q}$ this is the same as first representing $\mathbf{x}$ by (x_i), where $x_i \in \mathbb{Z}$ and $\gcd(x_1, \ldots, x_n) = 1$, and defining $H(\mathbf{x}) = \max_{0 \leqslant i \leqslant n} |x_i|$. The following is a standard fact about heights (Theorem 5.11 in [32]).

Lemma 6.4 *For any C, the set $\{\mathbf{x} \in \mathbf{P}^n(K) : H(\mathbf{x}) \leqslant C\}$ is finite.*

For an elliptic curve and the Jacobian of a curve of genus 2, one can define a function using the $L(2\Theta)$ embedding of the Kummer variety.

Definition 6.5 *Let $\mathcal{E}$ be an elliptic curve, as in (2.1). Define the* Kummer height *H_κ on $\mathcal{E}$ by $H_\kappa(P) = H(\kappa_0(P), \kappa_1(P))$, where the κ_i are as in (2.3). Let $\mathcal{C}$ be a curve of genus 2, as in (2.6), with Jacobian J. Define the* Kummer height *H_κ on $J(K)$ by $H_\kappa(D) = H(\kappa_0(D), \ldots, \kappa_3(D))$, where the κ_i are as in (2.8).*

It is immediate from Lemma 6.4 that H_κ always satisfies axiom (3) of Definition 6.1. For elliptic curves, this height is sometimes called the *x-coordinate height*. In both genus 1 and 2, there are polynomials B_{ij} biquadratic in the $\kappa_j(P)$, $\kappa_j(Q)$ and defined over K such that

$$\big(\kappa_i(P+Q)\kappa_j(P-Q) + \kappa_i(P-Q)\kappa_j(P+Q)\big) = 2\big(B_{ij}(P,Q)\big). \tag{6.1}$$

For elliptic curves these are given on p.416 of [32]. In the genus 2 case, see p.23 of [8]. These biquadratic forms easily give a constant C_1, dependent only on the coefficients of the B_{ij}, such that axiom (1) of Definition 6.1 is satisfied. Furthermore, setting $A = B$ gives the $\big(\kappa_i(2A)\big)$ in terms of non-degenerate quartics. Combining this with Hilbert's Nullstellensatz gives a constant C_2 for which axiom (2) of Definition 6.1 is satisfied. We may therefore deduce the following.

Theorem 6.6 *Let K, $\mathcal{E}$, $\mathcal{C}$, J, H_κ be as in Definition 6.5. Then the H_κ defined on $\mathcal{E}(K)$ in Definition 6.5 is a height function. Similarly, the H_κ defined on $J(K)$ in Definition 6.5 is also a height function.*

It follows that, for elliptic curves and Jacobians of curves in genus 2 we have an effective procedure for going from a description of $\mathcal{E}(K)/2\mathcal{E}(K)$ or $J(K)/2J(K)$ (as can sometimes be obtained by a 2-descent) to a set of generators for the Mordell-Weil group $\mathcal{E}(K)$ or $J(K)$, respectively. Furthermore, the height constant C_2 can be improved by using additional structure on the duplication law. One first notes that addition by a point of order 2 induces a linear map on (κ_i), both in genus 1 and 2 (see Chapter 12 of [8]). Choosing a set of 2^g points of order 2 as the kernel of a 2^g-isogeny, one can simultaneously diagonalise the corresponding linear maps. If one does this also to the dual isogeny, it is possible to express the duplication map on the Kummer as

$$[2] = M_1 \tau M_2 \tau M_3, \tag{6.2}$$

where M_1, M_2, M_3 are matrices defined over some field extension L, and $\tau : (\kappa_i) \mapsto (\kappa_i^2)$. A height constant C_2 can then be derived from the entries of these matrices, which improves on the height constant derived directly from applying Hilbert's Nullstellensatz to the quartic forms.

It is natural to ask whether the same can be attempted in genus 3, using the embedding we have found for the Kummer variety.

Definition 6.7 *Let $\mathcal{C}$ be a hyperelliptic curve of genus 3, as in (1.1) with $n = 7$, with Jacobian J. Define the* Kummer height H_κ *on $J(K)$ by $H_\kappa(D) = H(\kappa_0(D), \ldots, \kappa_7(D))$, where the κ_i are as in Theorem 3.13.*

By analogy with the genus 1 and 2 situations, it may be possible to examine the morphism $\kappa^{(2)} \to \kappa^{(2)}$ (where $\kappa^{(2)}$ is the symmetric square of the Kummer variety) that is given by mapping the unordered pair $\{\pm P, \pm Q\}$ to $\{\pm(P+Q), \pm(P-Q)\}$, and deduce biquadratic forms analogous to those of (6.1), and so non-degenerate quartic forms for the duplication law. Furthermore, Lemma 2,4,7 and Proposition 6.1.1 of [25] suggest that addition by a point of order 2 should induce a linear map on the Kummer variety. If there are such objects, then the following would be true, by imitating the arguments for genus 1 and 2.

Conjecture 6.8 *Let $\mathcal{C}$, J, H_κ be as in Definition 6.7. Then the H_κ defined on $J(K)$ in Definition 6.7 is a height function.*

It would be of interest to find explicitly the corresponding height constants C_1, C_2 since these would allow (for the first time) the actual computation of generators for $J(K)$ for those examples where $J(K)/2J(K)$ has already been found.

A further benefit of finding C_2 explicitly is the following immediate consequence of axiom (2).

Corollary 6.9 *Let H be a height function on an Abelian group A. Any $P \in A$ of finite order satisfies $H(A) \leq C_2^{1/3}$.*

In view of axiom (3), it follows that any Abelian group with a height function has a finite torsion subgroup. If, in our case, we could find in genus 3 an explicit expression for C_2 in terms of the coefficients of the curve $\mathcal{C}$ (as already exists in genus 1 and 2) then we would automatically have an effective procedure for computing the torsion subgroup of $J(K)$.

A further target for the future would be to understand how to embed the Kummer variety for genus 3 curves of the form $y^2 = F(x)$, where $F(x)$ has degree 8. As already remarked in Section 3.8, the problem is in understanding the representation of divisor classes by sets of points, using maps only defined over the ground field. If we set aside this problem, and formally apply our usual algorithm to $y^2 = f_8x^8 + \ldots + f_0$, we obtain variants of the c_i in (3.13).

$$
\begin{aligned}
c_0' &= c_0 + f_8(2a_2a_1^2 - a_1^4 - a_2^2),\\
c_1' &= c_1 + f_8(4a_2a_1^3 - a_1^5 - 3a_2^2a_1 - 2a_3a_1^2 + 2a_3a_2^3),\\
c_2' &= c_2 + f_8(a_2a_1^4 - 3a_2^2a_1^2 + a_2^3 + 2a_3a_2a_1 - a_3^2),\\
c_3' &= c_3 + f_8(a_2^2a_1^3 - a_3a_1^4 - 2a_2^3a_1 + 2a_3a_2a_1^2 + a_3a_2^2 - a_3^2a_1).
\end{aligned}
$$

What is interesting is that these still have o_n equal to 2, hinting at the possibility of describing the Kummer variety even in this case. It would also be interesting to try to describe the Kummer variety for the case where a genus 3 curve is a smooth plane quartic (every genus 3 curve is either hyperelliptic or a smooth plane quartic), but we have no idea how to do this.

For hyperelliptic curves of genus 4, there are no problems with representing divisor classes by sets of points, using maps only defined over the ground field since, as in genus 2, we have available $(g/2)\Theta^+ + (g/2)\Theta^-$. In principle, our ideas should be able to describe the Kummer variety for both the cases where $F(x)$ has degree 9 and degree 10. In particular, it is straightforward to check that the reduction of b_0^2 modulo δ^2 gives us a new basis member. The difficulty in genus 4 is primarily the size of the expressions involved and the number of monomials to be checked.

6.1 Addition on the Jacobian of a curve of genus g

Cassels [7] recites well-known formulæ for explicitly expressing the addition of points on the Jacobian of an elliptic curve in terms of points on the original curve; Cassels and Flynn [8] give an equivalent method in Chapter 8 for curves of genus 2. Generalizing the approach may be a useful tool for future work.

Given the curve $\mathcal{C} : y^2 = F(x)$, with $\partial F = 2g+1$ and the divisor $D = \sum m_i P_i$ s.t. $P_i = (x_i, y_i)$, $x_i = x_j$ iff $i = j$, and define $d = 2g - \deg D \geq 0$; we seek the curve $\mathcal{D} : q(x)y = p(x)$, $\partial q \leq \lfloor \frac{g-d-1}{2} \rfloor$ and $\partial p \leq \lfloor \frac{3g-d}{2} \rfloor$, s.t., for each P_i and $0 \leq n < m_i$,

$$\left. \frac{d^n}{dx^n} y_{\mathcal{C}} \right|_{P_i} = \left. \frac{d^n}{dx^n} y_{\mathcal{D}} \right|_{P_i} . \tag{6.3}$$

Note that the L.H.S. of the above can be expressed purely in terms of y_i and the values of the first n derivatives of F evaluated at x_i, which are relatively easy to calculate without recourse to methods of symbolic computation.

We find ourselves in a situation with $\deg D = 2g - d$ equations linear in $\lfloor \frac{g-d-1}{2} \rfloor + \lfloor \frac{3g-d}{2} \rfloor + 1 = 2g - d$ unknowns (the coefficients of x^k in p and q). The curve $\mathcal{D}$ can then be recovered using simple linear algebra. In fact, we can simplify the calculation still further by removing the dependency on using a symbolic maths package to calculate the derivatives of y^2 wrt x.

Definition 6.10 *Define the rational polynomial h_k in $k+1$ variables $d_0, \ldots, d_k$*

as follows:

$$h_k(d_0,\dots,d_k)=\begin{cases} d_0 & k=0\\ \frac{d_1}{2d_0} & k=1\\ \frac{1}{2d_0}\left(d_k-\sum_{j=1}^{k-1}\binom{k}{j}h_jh_{k-j}\right) & k\geq 2\end{cases} \tag{6.4}$$

Theorem 6.11 *Given a function $P(X)$, n-times differentiable, then*

$$P^{(k)}(x)=h_k(d_0,d_1,d_2,\dots,d_k)$$

where $0\leq k\leq n$, $d_0=P(x)$, and d_i is the ith derivative of $P(X)^2$ evaluated at x.

Proof The result is obviously true for $k=1$, so let us assume that it is true for all $k<r<n$. It can be shown that for all $s\leq n$

$$\frac{d^s}{dX^s}\left[P(X)^2\right]=\sum_{i=0}^{s}\binom{s}{i}P^{(i)}(X)P^{(s-i)}(X)$$

Hence

$$\begin{aligned} P^{(r)}(x)&=\frac{1}{2d_0}\left(\frac{d^r}{dX^r}\left[P(X)^2\right]_{X=x}-\sum_{j=1}^{r-1}\binom{r}{j}P^{(j)}(x)P^{(r-j)}(x)\right)\\ &=\frac{1}{2d_0}\left(d_r-\sum_{j=1}^{r-1}\binom{r}{j}h_jh_{r-j}\right)\end{aligned}$$

as required. □

Definition 6.12 *Given a list of $n+1$ ordered pairs $(x_0,m_0),\dots,(x_n,m_n)$, define $N=\sum m_i$. Then take the N monomials $1,x,x^2,\dots,x^{N-1}$ and define the $N\times N$ matrix $\mathfrak{V}\big(\{(x_i,m_i)\}\big)$ by placing in the first row and ith column the value of the monomial x^{i-1} at x_0, into the second row and ith column the value of the first derivative of the monomial x^{i-1} at x_0, and so on up until the m_0th row wherein the value of the (m_0-1)th derivative of x^{i-1} at $x=x_0$ is placed into the ith column. Repeat this process in the following rows for each of the x_i.*

Example 6.13

$$\mathfrak{V}(\{(x_0,3),(x_1,4),(x_2,1)\}) = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & x_0^4 & x_0^5 & x_0^6 & x_0^7 \\ 0 & 1 & 2x_0 & 3x_0^2 & 4x_0^3 & 5x_0^4 & 6x_0^5 & 7x_0^6 \\ 0 & 0 & 2 & 6x_0 & 12x_0^2 & 20x_0^3 & 30x_0^4 & 42x_0^5 \\ 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 & x_1^6 & x_1^7 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 & 5x_1^4 & 6x_1^5 & 7x_1^6 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 & 20x_1^3 & 30x_1^4 & 42x_1^5 \\ 0 & 0 & 0 & 6 & 24x_1 & 60x_1^2 & 120x_1^3 & 210x_1^4 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 & x_2^6 & x_2^7 \end{bmatrix}$$

Theorem 6.14

$$\det \mathfrak{V}(\{(x_i,m_i)\}) = \prod_{0\le i<j\le n} (x_j - x_i)^{m_i m_j} \cdot \prod_{i=0}^{n} \prod_{k=0}^{m_i-1} k!$$

Proof See Proposition 4.2.1 in [20]. $\square$

Note that in the case where $m_i = 1$ for every i, we have the usual Vandermonde determinant of the x_i's.

Example 6.15

$$\det \mathfrak{V}(\{(x_0,3),(x_1,4),(x_2,1)\}) = 24(x_2-x_1)^4(x_1-x_0)^{12}(x_2-x_0)^3$$

Theorem 6.16 *Given a hyperelliptic curve $\mathcal{C} : Y^2 = F(X)$ and a list (x_i, y_i) of n distinct points each with multiplicity m_i s.t. $x_i \neq x_j$ whenever $i \neq j$. Suppose that $d = \left(\sum m_i\right) - 1$. Then there is a unique polynomial $G(X)$ of degree d s.t. the locus $\mathcal{D} : Y = G(X)$ passes through each of the points (x_i, y_i) and, further, that for all $1 \le k \le m_i - 1$*

$$\frac{d^k}{dX^k}\Big[G(X)^2\Big]_{X=x_i} = F^{(k)}(x_i)$$

Proof With the notation h_k from Theorem 6.11 we note that we have the $d+1$ equations

$$G(x_i) = h_0(y_i) \tag{6.5}$$

$$G^{(k)}(x_i) = h_k\big(y_i, F^{(1)}(x_i), F^{(2)}(x_i), \cdots, F^{(k)}(x_i)\big) \tag{6.6}$$

Our system of defining equations is in fact a system of $(d+1)$ inhomogeneous linear equations in the $(d+1)$ coefficients g_i in $G(X) = g_d + g_{d-1}X + \cdots + g_0X^d$. Define $r(i) = (1 + \sum_{j=0}^{i-1} m_j)$, then

$$\Big[\mathfrak{V}(\{(x_0, m_0), \cdots, (x_n, m_n)\}) \;\;\Big|\;\; H\Big] \tag{S}$$

where H is the $(d+1) \times 1$ matrix with row $k + r(i)$ $(0 \le k < m_i)$ is

$$\Big[h_k\big(y_i, F^{(1)}(x_i), F^{(2)}(x_i), \cdots, F^{(k)}(x_i)\big)\Big].$$

In order to show that we have a solution it is necessary to show that the rowspace of our system (S) is equal to that of $\mathfrak{V}$. For this solution to be unique we need these both in turn to equal $d+1$. In this respect it is necessary and sufficient to show that the rowspace of $\mathfrak{V}$ is $d+1$, i.e. that the determinant $\det \mathfrak{V}$ is nonzero.

Theorem 6.14 implies that $\det \mathfrak{V} = 0$ iff $x_i = x_j$ for distinct i, j, but this does not happen in our list of points, so the determinant is nonzero as required. $\square$

Example 6.17 *Consider the genus 2 curve* $\mathcal{C} : Y^2 = F(X)$ *where*

$$F(X) = X(X-1)(X-2)(X-5)(X-6)$$

and the list of points $(0,0)$, $(10,-120)$, $(3,6)$, $(3,6)$.

$$\mathfrak{V}(\{(0,1),(10,1),(3,2)\}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 \\ 1 & 3 & 9 & 27 \\ 0 & 1 & 6 & 27 \end{bmatrix} \qquad H = \begin{bmatrix} 0 \\ -120 \\ 6 \\ 3 \end{bmatrix}$$

We obtain the system

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 100 & 1000 & -120 \\ 1 & 3 & 9 & 27 & 6 \\ 0 & 1 & 6 & 27 & 3 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & \frac{7}{3} \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \end{array}\right]$$

So define $G(X) = (0) + (-2)X + (\frac{7}{3})X^2 + (-\frac{1}{3})X^3$. *G satisfies:*

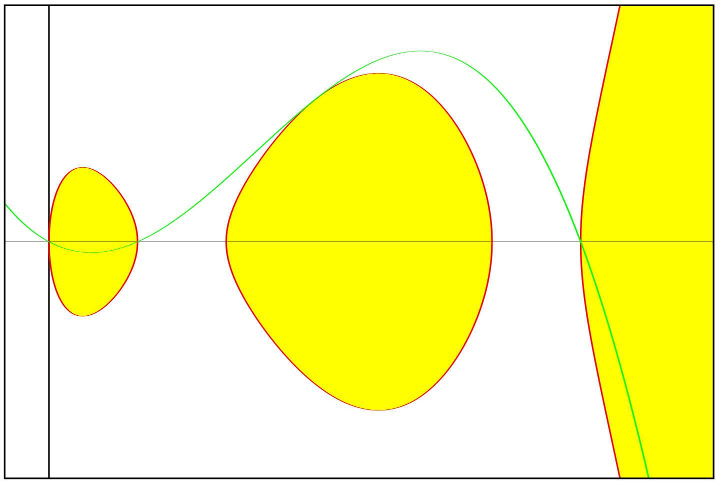

Figure 6.1: Addition from 6.7

(i). $G(0) = 0,\ G(3) = 6,\ G(10) = -120$

(ii). $\frac{d}{dX}\left[G(X)^2\right]_{X=3} = 36 = F^{(1)}(3)$

(iii). $G(X)^2 - F(X) = \frac{1}{9}X(X-1)(X-3)^2(X-6)(X-10)$

We can deduce, as shown in the figure 6.1, that

$$\{(0,0),\ (10,-120)\} + \{(3,6),\ (3,6)\} = \{(1,0),\ (6,0)\} \tag{6.7}$$

6.2 Overview of Addition Algorithm

The algorithm is basically an implementation of calculating $\mathfrak{V}$ and H and then solving the system S. We then need to factor $G(X)^2 - F(X)$, but our task is simplified by the fact that we already know many of the factors. In the genus 2 case, with 4 points (counting multiplicity), we have at most a quadratic to solve.

Before we plug our points into the algorithm, we need to make some simplifications for 'special cases'.

- $\{A, B\} + \{C, D\} = \{A, C\} + \{B, D\} = \{A, D\} + \{B, C\}$ etc.
- When $Y^2 = F(X)$, and the degree of F is 5
  (i). $\{\infty, \infty\}$ and $\{(x, y), (x, -y)\}$ are all identified as $\mathfrak{O}$ s.t. $\mathfrak{A} + \mathfrak{O} = \mathfrak{A}$.
- When $Y^2 = F(X)$, and the degree of F is 6
  (i). We now have points ∞^+, ∞^-, so that $\{\infty^+, \infty^-\}$ and $\{(x, y), (x, -y)\}$ are all identified as $\mathfrak{O}$.
  (ii). The occurrence of ∞^+ or ∞^- produces restraints on the possible $G(X)$ which cannot be represented in the system from the theorem except by birationally transforming the curve.
- We drop occurrences of $\mathfrak{O}$ from our list of points, hence we need either 3 or 4 distinct points satisfying the hypothesis of the theorem or else we can return a result simply by rearranging the points we started with.

The above algorithm, using $\mathfrak{V}$ is sufficient for genus 2, but for higher genus the degree of G is such that we have too many roots; in this situation, we must solve for p and q. The system of equations is given by:

$$\frac{d^n}{dx^n}\big(q(x)y\big)\bigg|_{P_i} = \frac{d^n}{dx^n}p(x)\bigg|_{P_i} \tag{6.8}$$

and the coefficients of p and q are given by:

$$p_k : \begin{cases} 0 & k < n \\ \frac{k!}{(k-n)!}x^{k-n} & n \leq k \leq \partial p \end{cases} \tag{6.9}$$

$$q_k : \sum_{i=0}^{k} \binom{n}{i} \frac{k!}{(k-i)!} x^{k-i} \frac{d^{n-i}y}{dx^{n-i}} \qquad 0 \leq k \leq \partial q \tag{6.10}$$

Appendix A

Summary of Key Programs

We shall summarise a few of the main files of Maple programs used in our computations. Even though they are designed for general hyperelliptic curves of genus g (with an emphasis on genus 3), they have nevertheless been placed in a subdirectory of E.V. Flynn's genus 2 ftp directory, namely:

`ftp://ftp.liv.ac.uk/pub/genus2/stubbs`

The files are:

checkquarts.txt

This checks lists the 27 relations of Theorem 5.2, and checks that they are identities after expanding and replacing each y_i^2 by $F(x_i)$. Note that the expressions are far too large to expand initially, and so the file slices the expressions into pieces and simplifies a portion at a time.

fast_mult.txt

This gives a fast way of computing the coefficient of $y_1 y_2$ in a product of the form $c_i c_j$, where $c_0, \ldots, c_3$ are as in (3.13). This is helpful for finding the relations in Theorem 5.2.

getkumm.txt

This runs through monomials of the form $a_i b_j b_k$, finds all linear combinations (four of them) for which the coefficient of $y_1 y_2$ has o_n at most 2, and hence finds and outputs $c_0, \ldots, c_3$ of (3.13).

gpolyutil.txt

This inputs the files mvmoreutils.txt, mvutils.txt, symm.txt, wreduce.txt, and hence loads the package I have written for general routines for hyperelliptic curves (with an emphasis on genus 3). The routines which become available after the package is loaded dividing between these four files, and are as follows.

checkonesym, conv, divalg, g3onesymmp, g3weight, get_heaviest_terms, get_null_space, get_orbit, get_weight_greaterthan, gethigh, initf, lc, lm, multideg, mvcoeff, mvexps, mvmax, mvmono, mvnorm, mvorder, naive_reduce_g3, onesym, onesymm, onesymminit, permute_vars, simplify_poly_table_proc, try_simple_reduce_g3

mvmoreutils.txt

These are multivariate polynomial algorithms, such as get_null_space, calculate_hgs_part, which extract parts of highest weight, and which compute the space on induced by the requirement that linear combinations of a collection of expressions should have highest weight portion set to zero.

mvutils.txt

General purpose routines for extracting portions of information, such as extracting specified monomials, and division algorithms.

symm.txt

Programs for expressing symmetric functions as polynomials of elementary symmetric functions.

symmvar.txt

Same as symm.txt, but with $a_0, \ldots, a_g$ being used as the choice for the elementary symmetric functions, and with the output being made homogeneous.

wreduce.txt

The program naive_reduce_g3 performs reduction modulo a power of delta, and g3weight computes the weight o_n of Definition 3.2.

hell/

This directory contains files containing algorithms used in calculating the c_i, as discussed in section (3.9). Whilst many of the algorithms are available in files not contained in this directory, the nature of the unpredictable interactions between slightly different versions of the algorithms I experienced in using `maple` (especially in algorithms designed to work for slightly different versions of `maple` itself) suggests it is wiser to use files that are known to work together. I have verified that the files given in this directory can be used together to produce the c_i for genus 3.

hell/thesis.v5

This is a convenience file to read in the other files in this directory into `maple`.

hell/hell.v5

This contains the major ('front end') algorithms used in producing the c_i. Of particular interest in this file are the functions `get_monos`, which, given the degree of F, and the required w- and y-weights, returns a list of monomials in the a_i, b_i, and f_i which sit in that weight partition; `distillation`, which, given the degree of F and a list of monomials, returns an array of polynomials sorted into levels of the order of the pole on Θ, after a distillation has been performed; `get_nTheta` performs a triple-pass distillation given the genus of the curve, the w- and y-weight partition to search for polynomials, and the order of the pole on Θ that we want our return polynomials to have.

hell/mvutil.v5

A hybrid of the functions available in files in the parent directory.

hell/pairsym.v5

Some initial work in symmetric polynomials in pairs of variables.

hell/sym.v5

More work in symmetric polynomials.

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