Mathematics and the environment

PROBLEM SHEET 1.

1. The planetary albedos of Venus, Mars and Jupiter are 0.77, 0.15, 0.58 respectively, and their distances from the sun are 0.72, 1.52, 5.20 *astronomical units* (1 a.u. = distance from Earth to the sun). Calculate the radiative equilibrium temperature of these planets, and compare them with the measured effective black body temperatures, T_m = 230 K, 220 K, 130 K (i. e., the temperature corresponding to the amount of long wave radiation actually emitted). Which, if any, planets appear not to be in equilibrium; can you think why this might be so?

   [*The albedo of the Earth is 0.3 and its radiative equilibrium temperature is 255 K, and the received planetary short wave radiation varies inversely with the square of distance from the sun.*]

2. Explain what is meant by the two stream approximation to the radiative transfer equation, given by

$$\begin{aligned} \tfrac{1}{2}I'_+ &= -\kappa\rho\,[I_+ - B]\,, \\ -\tfrac{1}{2}I'_- &= -\kappa\rho\,[I_- - B]\,, \end{aligned}$$

   where $B = \frac{1}{2}(I_+ + I_-) = \sigma T_a^4/\pi$, and T_a is the air temperature. What are appropriate boundary conditions for these equations? If the upwards and downwards fluxes $F_\pm$ are given by $F_\pm = \pi I_\pm$, show that the net upwards flux $F_+ - F_- = \Phi$ is conserved, and hence give the solutions for $I_\pm$ in terms of this flux and the optical depth $\tau = \displaystyle\int_z^h \kappa\rho\,dz$.

   Deduce that the planetary surface temperature T_s is higher than the effective long wave emission temperature T_e, and show that it increases with the optical depth $\tau_s = \displaystyle\int_0^h \kappa\rho\,dz$ of the atmosphere (where h is the depth of the atmosphere).

   Deduce also that the surface air temperature $T_{as} = T_a\big|_{z=0}$ is lower than the ground surface temperature T_s.

3. A wet adiabat is calculated from the isentropic equation

$$\rho_a c_p \frac{dT}{dz} - \frac{dp}{dz} + \rho_a L \frac{dm}{dz} = 0,$$

   where

$$m = \frac{\rho_v}{\rho_a}, \; p = \frac{\rho_a RT}{M_a}, \; p_{SV} = \frac{\rho_v RT}{M_v},$$

   and

$$\frac{dp_{SV}}{dT} = \frac{\rho_v L}{T}, \; \frac{dp}{dz} = -\rho_a g.$$

Deduce that T and p_{SV} can be calculated from the equations

$$\begin{aligned}\frac{dT}{dz} &= -\Gamma_w(\rho_v, p, T),\\ \frac{dp_{SV}}{dz} &= -\frac{\rho_v L}{T}\Gamma_w,\end{aligned}$$

where $\rho_v = \rho_v(p_{SV}, T)$, and Γ_w should be determined. Using values $M_v/M_a = 0.62$, $L = 2.5 \times 10^6$ J kg^{-1}, $T = 290$ K, $c_p = 10^3$ J kg^{-1} K^{-1}, $\rho_v = 0.01$ kg m^{-3}, $p = 10^5$ Pa, $g = 10$ m s^{-2}, $\rho_a = 1$ kg m^{-3}, show that a typical value of Γ_w is 6 K km^{-1}.

By assuming that $T \approx$ constant near the ground, show that p_{SV} satisfies the differential equation

$$\frac{dp_{SV}}{dp} = \frac{\beta p_{SV}}{p}\left[\frac{1+\dfrac{ap_{SV}}{p}}{1+\dfrac{\beta a p_{SV}}{p}}\right],$$

where the two dimensionless coefficients a and β are given by

$$a = \frac{M_v L}{RT}, \quad \beta = \frac{M_v}{M_a}\frac{L}{c_p T}.$$

Estimate the values of a and β (you will need also the values $M_v = 18 \times 10^{-3}$ kg mole^{-1}, $R = 8.3$ J mole^{-1} K^{-1}). Hence show that the *molar humidity* $h = p_{SV}/p$ decreases approximately linearly with altitude. (*The other commonly used quantity is the relative humidity,* p_v/p_{SV}.)

Mathematics and the environment

PROBLEM SHEET 2.

1. Show that the solution of the Clausius-Clapeyron equation for saturation vapour pressure p_{SV} as a function of temperature T is

$$p_{SV} = p_{SV}^0 \exp\left[a\left\{1 - \frac{T_0}{T}\right\}\right],$$

where for water vapour, we may take $T_0 = 273$ K at $p_{SV}^0 = 6$ mbar ($= 600$ Pa), the *triple point*, and $a = M_v L/RT_0$. Show that if T is close to T_0, then

$$p_{SV} \approx p_{SV}^0 \exp\left[a\left(\frac{T - T_0}{T_0}\right)\right].$$

If the long wave radiation from a planet is $\sigma\gamma T^4$, where T is the mean surface temperature, if the solar flux is Q (and planetary albedo is zero), and the greyness factor is taken to be given by

$$\gamma^{-1/4} = 1 + b(p_v/p_{SV}^0)^c,$$

where p_v is the H_2O vapour pressure, show that the occurrence of a runaway greenhouse effect is controlled by the intersection of the two curves

$$\theta = 1 + \lambda\xi, \;\; \theta = \rho(1 + be^{c\xi}),$$

where $\lambda = 1/a$, $\rho = (Q/4\sigma T_0^4)^{1/4}$. Show (for example, by considering the graphs of the two functions θ of ξ defined above) that runaway occurs if $\rho > \rho_c$, where

$$\rho_c + \delta = 1 + \delta\ln[\delta/b\rho_c]$$

with $\delta = \lambda/c$. Show that this determines a unique value of ρ_c, and that if δ is small,

$$\rho_c \approx 1 + \delta\ln(\delta/b) - \delta.$$

Estimate values of ρ and λ appropriate to the present Earth, and comment on the implications of these values for climatic evolution if we choose $b = 0.06$, $c = 1/4$. What are the implications for Venus, if the solar flux is twice as great? What if solar radiation were 30% lower when the planetary atmospheres were being formed?

2. For the energy balance model

$$c\dot{T} = R_i - R_o,$$

where $R_i = \frac{1}{4}Q(1-a)$, $R_o = \sigma\gamma T^4$, and $a = a_+$ for $T < T_i$, $a = a_-$ for $T > T_w$ ($> T_i$), $a_+ > a_-$, with $a(T)$ linear between these two ranges, show that possible steady state values of T are $T = T_i$ when $Q = Q_+$ and $T = T_w$ when $Q = Q_-$, where

$$Q_- = \frac{4\sigma\gamma T_w^4}{1-a_-}, \quad Q_+ = \frac{4\sigma\gamma T_i^4}{1-a_+}.$$

By considering the graphs of R_o and R_i, and the slope of $R_o(T)$ at T_i, show that for Q just less than Q_+, multiple steady states will occur if

$$\frac{T_w - T_i}{T_i} < \frac{a_+ - a_-}{4(1-a_+)},$$

and in this case show that they will exist in a range $Q_c < Q < Q_+$, and prove that the upper and lower branches are stable, but the intermediate one is unstable.

By considering the slope of $R_0(T)$ at T_w, show that if

$$\frac{T_w - T_i}{T_w} < \frac{a_+ - a_-}{4(1-a_-)},$$

then $Q_c = Q_-$.

By normalising Q and T with respect to present day values Q_0, T_0 satisfying $Q_0(1-a_-) = 4\sigma\gamma T_0^4$, show that the corresponding dimensionless solar fluxes and mean atmospheric temperatures q and θ satisfy

$$\begin{aligned} q_- &= \theta_w^4, \\ q_+ &= \theta_i^4 \left(\frac{1-a_-}{1-a_+}\right), \end{aligned}$$

and that multiple steady states will occur providing

$$\frac{\theta_w - \theta_i}{\theta_w} < \frac{a_+ - a_-}{4(1-a_-)}.$$

If $\theta_w = 1$ (we are starting an ice age *now*) show that if $\theta_i = 1 - \delta$, $a_+ = a_- + \nu$, where $\delta, \nu \ll 1$, then regular ice ages will occur providing

$$\delta < \frac{\nu}{4(1-a_-)},$$

and providing the solar flux q oscillates beyond the limits $q_+ \approx 1 + \nu/(1-a_-) - 4\delta$ and $q_- = 1$.

Mathematics and the environment

PROBLEM SHEET 3.

1. Find a relationship between the hydraulic radius R and the area A for triangular (notch shaped) canals of transverse bedslope θ and for rectangular (canal shaped) cross sections of width w. Hence show that Chézy's and Manning's laws both lead to a general relationship of the form

$$Q = \frac{cA^{m+1}}{m+1},$$

with $0 < m < 1$, giving explicit prescriptions for c and m. (*You should assume that the rectangular channel is very wide.*) For a canal of depth h, show that the flow is turbulent if

$$h \gtrsim 10^2 \nu^{2/3} \left(\frac{f}{Sg}\right)^{1/3},$$

where ν is the kinematic viscosity, f is the friction factor, S is the slope and g is gravity. (*Take the Reynolds number to be uh/ν, and use Chezy's law for the velocity.* Taking $\nu = 10^{-6}$ m^2 s^{-1}, $f = 0.01$, $S = 10^{-3}$, $g = 10$ m s^{-2}, find a critical depth for turbulence. Is the Isis turbulent?

2. For flow in a pipe, the friction factor f in the formula $\tau = f\rho u^2$ is often taken to depend on the Reynolds number, for example, Blasius's law of friction has $f \propto Re^{-1/7}$. By taking $Re = uR/\nu$, where R is the hydraulic radius, find modifications to Chézy's law if $f \propto Re^{-\beta}$. Comment on whether you can obtain Manning's flow law this way.

3. The cross sectional area of a river A is assumed to satisfy the wave equation

$$\frac{\partial A}{\partial t} + cA^m \frac{\partial A}{\partial s} = 0,$$

where s is distance downstream. Explain how this equation can be derived from the principle of conservation of mass. What assumptions does your derivation use?

A river admits a steady discharge $Q = Q_+$. At $t = 0$, a tributary at $s = 0$ is blocked, causing a sudden drop in discharge to $Q_- < Q_+$. Solve the equation for A using a characteristic diagram and show that an *expansion fan* branches from $s = 0$, $t = 0$. What is the hydrograph record at a downstream station $s = s_0 > 0$?

Later, the tributary is re-opened, causing a sudden rise from Q_- to Q_+. Draw the characteristic diagram, and show that a shock wave propagates forwards. What is its speed?

4. The equation describing slowly-varying river flow is

$$\frac{\partial A}{\partial t} + cA^m \frac{\partial A}{\partial s} = 0,$$

where A is the cross-sectional area, s is distance downstream, and c and m are positive and constant. Describe what this equation means, and give an expression (in terms of A) for the mean stream velocity u.

For a general initial condition

$$A = A_0(s) \quad \text{at} \quad t = 0, \quad -\infty < s < \infty,$$

use the method of characteristics to find the general solution of the model. Show also that in general shocks will form, and describe in what situations they will not. What happens in the latter case?

Either by consideration of an integral form of the conservation of mass equation, or by consideration from first principles, derive a jump condition which describes the shock speed. In terms of the stream speed u, what is the speed of a shock (a) when it first forms; (b) when it advances over a dry river bed (as in a flash flood)?

Mathematics and the environment

PROBLEM SHEET 4.

1. Derive the St. Venant equations from first principles in the form

$$u_t + uu_s = gS - \frac{\tau l}{\rho A} - g\frac{\partial \bar{h}}{\partial s},$$

$$A_t + (Au)_s = 0,$$

indicating the meaning of the various terms.

Derive the form of these equations assuming that Manning's roughness law

$$u = \frac{R^{2/3}S^{1/2}}{n'}$$

describes steady, uniform flow, where R is the hydraulic radius, and that the stream has triangular cross section, of transverse bed angle θ.

Show how to non-dimensionalise the equations to obtain

$$A_t + (Au)_s = 0,$$

$$\varepsilon F^2(u_t + uu_s) = 1 - \frac{u^2}{A^{2/3}} - \varepsilon\frac{\partial}{\partial s}\left(A^{1/2}\right),$$

and define the parameters ε and F.

A sluice gate on the river is suddenly opened so that the (dimensionless) discharge there increases from Q_- to Q_+. The hydrograph is measured a distance l downstream. Using l as the length scale in the non-dimensionalisation above, derive an approximate expression for the dimensionless discharge in terms of A, if the Froude number F is small, and also $\varepsilon \ll 1$.

Hence show that A satisfies the approximate equation

$$\frac{\partial A}{\partial t} + \tfrac{4}{3}A^{1/3}\frac{\partial A}{\partial s} = \tfrac{1}{4}\varepsilon\frac{\partial}{\partial s}\left[A^{5/6}\frac{\partial A}{\partial s}\right].$$

What do you think the difference between the hydrographs for $\varepsilon = 0$ and $0 < \varepsilon \ll 1$ might be?

2. Using Chézy's law with a rectangular cross section, derive an appropriate form of the St. Venant equations, and show how to non-dimensionalise them to obtain the equations in the form

$$A_t + (Au)_s = 0,$$

$$F^2(u_t + uu_s) = 1 - \frac{u^2}{A} - A_s.$$

Give the definition of the Froude number F, and show that the length scale $s^* = h^*/S$, where h^* is the depth scale, and S is the bedslope..

Choose or guess suitable values for the Thames in London, the Isis/Cherwell in Oxford, an Alpine (or other) mountain stream, and determine the corresponding natural length and time scales, and the Froude number, for these flows.

Long wave theory describes solutions varying on a scale s^*/ε, where $\varepsilon \ll 1$, while short wave theory describes solutions varying on a scale δs^*, where $\delta \ll 1$. By suitably rescaling s and t in the equations, show also in these two cases, further approximations to the model can be made. (The equations effectively become those of slowly varying flow and the shallow water equations of B6, respectively.)

3. Use the dimensionless form (with $\varepsilon = 1$) of the St. Venant equations which assumes Manning's roughness law and a triangular river cross section (see question 1) to show that small disturbances to the steady state can propagate up and down stream if $F < F_1$, but can only propagate downstream if $F > F_1$, and that they are unstable if $F > F_2$. What are the values of F_1 and F_2?

4. *The hydraulic jump*

Show that the dimensionless form of the mass and momentum equations for a canal (see question 2) can be written in the conservation form

$$A_t + (Au)_s = 0,$$

$$F^2[(Au)_t + (Au^2)_s] = A - u^2 - (\tfrac{1}{2}A^2)_s.$$

Show that discontinuities (shocks) in the channel depth travel at a (dimensionless) speed V given by

$$V = \frac{[Au]_-^+}{[A]_-^+} = \frac{[F^2Au^2 + \frac{1}{2}A^2]_-^+}{[F^2Au]_-^+},$$

where $\pm$ refer to the values on either side of the jump, and F is the Froude number.

[*Both mass and momentum must be conserved across the shock, and from first principles one can show that the conservation law*

$$\phi_t + \boldsymbol{\nabla}.\mathbf{q} = \mathbf{c},$$

where $\mathbf{c}$ is algebraic, leads to the normal shock speed

$$V = \frac{[q_n]_-^+}{[\phi]_-^+}.]$$

Show that a stationary jump at $s = 0$ is possible (this can be seen when a tap is run into a basin) if $Au = Q$ in $s > 0$ and $s < 0$, and

$$\left[\frac{F^2Q^2}{A} + \frac{A^2}{2}\right]_-^+ = 0.$$

Deduce that for prescribed Q and A_-, a unique choice of $A_+ \neq A_-$ is possible.

The locally defined Froude number Fr is defined in terms of dimensional quantities as $Fr = \dfrac{u}{\sqrt{gh}}$. Show that in the present model, this implies

$$Fr = \frac{FQ}{A^{3/2}},$$

where Q and A are dimensionless flux and cross section, respectively.

Deduce that the hydraulic jump connects a region of *supercritical* ($Fr > 1$) flow to a *subcritical* ($Fr < 1$) one. (In practice, $A_- < A_+$ if $Q > 0$; if $A_- > A_+$, the discontinuity cannot be maintained.)

Mathematics and the environment

PROBLEM SHEET 5.

1. Write down the Exner equation for bedload transport, and show how it can be used to study the onset of bedform instability, assuming a suitable bedload transport law. Show that in conditions of slow flow, the resultant equation for the bed profile $s(x,t)$ is a first order hyperbolic equation, and deduce that the profile is neutrally stable. Show also that bed waves will form shocks which propagate downstream.

   Now suppose that the bedload transport $q_b(x,t)$ is a function of the basal stress τ evaluated at $x-\delta$. Show that instability can occur if $\delta < 0$, i.e. the stress *leads* the bed profile.

   Can you think of a physical reason why such a lead should occur?

   Do you think such a model would be a good *nonlinear* model?

2. The Exner equation for bed evolution is written in the form

$$(1-n)s_t + q_x = 0,$$

   and the bedload transport is given by

$$l_q q_x = [q_0(\tau) - q],$$

   where τ is the bed shear stress, and l_q is a constant length. The river flow is described by the quasi-static St. Venant equations,

$$uh = Q,$$

$$uu_x = g(S - \eta_x) - \frac{\tau}{\rho h},$$

   and $\eta = s + h$, $\tau = f\rho u^2$.

   Explain in physical terms why the equation for q should be appropriate to describe bedload transport.

   Show how to non-dimensionalise these equations to obtain the set

$$uh = 1,$$

$$F^2 uu_x = 1 - \eta_x - \frac{u^2}{h},$$

$$s_t + q_x = 0,$$

$$\delta q_x = q^*(\tau) - q,$$

   and determine the parameters δ and F.

Write down a suitable steady state solution, and show that if $q_0(\tau)$ is a monotonically increasing function of τ, then the steady state is linearly unstable to perturbations of dimensionless wavenumber k if

$$F > 1, \quad k > \left[\frac{3}{\delta(F^2-1)}\right]^{1/2}.$$

Show also that the corresponding waves move upstream. Show that the growth rate remains positive as the wavenumber $k \to \infty$. [*This is an indication of ill-posedness.*]

Mathematics and the environment

PROBLEM SHEET 6.

1. A St. Venant type model for river flow and bed erosion is given by

$$h_t + (uh)_x = 0,$$

$$u_t + uu_x = g(S - \eta_x) - \frac{\tau}{\rho_w h},$$

$$\frac{\partial}{\partial t}(hc) + \frac{\partial}{\partial x}(hcu) = \rho_s(v_E - v_D),$$

$$(1-n)\frac{\partial s}{\partial t} + \frac{\partial q_b}{\partial x} = -(v_E - v_D).$$

Define the meaning of the terms in these equations.

Parker (J. Fluid Mech. **89**, p. 109 (1978)) suggests the following expressions for the erosion and deposition rates in a stream:

$$v_E = bu_*^3/v_s^2, \quad v_D = \frac{\gamma v_s^2 c}{\rho_s u_*},$$

where c is the sediment concentration (mass per unit volume), v_s is the settling velocity, u_* is the 'friction' velocity $(\tau/\rho_w)^{1/2}$, and b and γ are constants (≈ 0.007 and 13, respectively). Suppose that $\tau = f\rho_w u^2$, and that bedload transport is given by

$$q_b = \frac{\rho_w K u_*^3}{\Delta\rho g},$$

where $K = 8$ and $\Delta\rho = \rho_s - \rho_w$.

Show how to non-dimensionalise the equations to obtain

$$h = \eta - s,$$

$$\varepsilon h_t + (uh)_x = 0,$$

$$F^2(\varepsilon u_t + uu_x) = -\eta_x + \delta\left(1 - \frac{u^2}{h}\right),$$

$$h(\varepsilon c_t + uc_x) = u^3 - \frac{c}{u} = -\left[s_t + \beta q_x\right],$$

where $q = u^3$, and define the parameters ε, F, δ and β. (Assume that the volume flux per unit stream width is q.)

Use values $g = 10$ m s^{-2}, $S = 10^{-3}$, $f = 0.05$, $\rho_s = 2.5 \times 10^3$ kg m^{-3}, $\rho_w = 10^3$ kg m^{-3}, and $v_s = 0.1$ m s^{-1}, to estimate the parameters.

Now suppose that $F, \varepsilon, \delta \ll 1$, but that $\beta = O(1)$. By examining the linear stability of the uniform state $c = u = h = 1$, show that the uniform state is stable for all values of β.

2. A simple model of bed erosion based on the St. Venant equations can be written in dimensionless form as

$$\varepsilon h_t + (uh)_x = 0,$$

$$F^2(\varepsilon u_t + uu_x) = -\eta_x + \delta\left(1 - \frac{u^2}{h}\right),$$

$$h(\varepsilon c_t + uc_x) = E^*(u) - c = -s_t,$$

where $h = \eta - s$. By considering the stability of the steady state $u = h = c = 1$ on a time scale t of $O(1)$, and assuming that $\delta \ll 1$, $\varepsilon \ll 1$, show that instability can occur depending on the size of $E^{*\prime}(1)$. Show also that η and s are out of phase if $F < 1$, and in phase if $F > 1$; interpret this in terms of dune and anti-dune formation.

3. * Write down the Exner equation relating bed profile s and bedload transport q_b, the Meyer-Peter/Müller bedload transport law relating q_b to basal stress τ, and suppose that $\tau = f\rho u^2$. Write down a suitable non-dimensional version of the model (in particular, so that, in dimensionless terms, $\tau = u^2$).

   It can be shown under certain assumptions that the existence of a non-constant bed profile $s(x,t)$ beneath a turbulent river flow leads to a modification of the basal stress which can be approximated by

$$\tau = u^2\left[1 - s + \int_0^\infty K(\xi)\frac{\partial s}{\partial x}(x - \xi, t)\, d\xi\right],$$

   where the kernel $K(x) \approx \mu/x^{1/3}$, and $\mu > 0$. Examine whether the modified model predicts (linear) instability in this case.

   In your calculation, you will need to evaluate the integral $\int_0^\infty \xi^{-1/3}e^{-ik\xi}\, d\xi$. Use contour integration to show that if $k > 0$, this integral is equal to

$$\frac{e^{-i\pi/3}}{k^{2/3}}\int_0^\infty t^{-1/3}e^{-t}\, dt,$$

   and thus that

$$\int_0^\infty \xi^{-1/3}e^{-ik\xi}\, d\xi = \frac{e^{-i\pi/3}\Gamma(\frac{2}{3})}{k^{2/3}},$$

   where $\Gamma(\nu)$ denotes the gamma function (see, for example, Abramowitz and Stegun, Handbook of mathematical functions, Dover 1968).

   What is the corresponding result if $k < 0$?

   [*The Fourier transform of the function $p(x) = x^{\nu-1}$, $x > 0$, $p(x) = 0$, $x < 0$, is a tabulated transform (e. g., F. Oberhettinger, Tables of Fourier transforms and Fourier transforms of distributions, Springer-Verlag 1990).*]

Mathematics and the environment

PROBLEM SHEET 7.

1. Use lubrication theory to derive an approximate model for two-dimensional flow of a valley glacier of depth H flowing over a bed $z = h(x)$, where (x, z) are cartesian coordinates along and transverse to the mean slope, assuming Glen's flow law with a rate constant independent of temperature, and no sliding at the base. Non-dimensionalise the model to the form

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left[\{1 - \mu(H_x + h_x)\}^n \frac{H^{n+2}}{n+2}\right] = a(x),$$

and show that for typical lengths of 10 km, accumulation rates of 1 m y^{-1}, and if the rate constant in Glen's law is 0.2 $\mathrm{bar}^{-3}\,\mathrm{y}^{-1}$ (with the Glen exponent being $n = 3$), a typical glacier depth is 100 m (note that 1 bar $= 10^5$ Pa). Show that $\mu = d \cot \alpha / l$, where d is the depth scale, l is the length scale, and α is the valley slope. What are typical values of μ?

Show that if $\mu \ll 1$, the model takes the form of a first order hyperbolic wave equation. Write down the steady state solution $H_0(x)$, and by linearising about this solution for small perturbations $H_1 = H - H_0$ to the steady state, show that the perturbations H_1 grow unboundedly near the glacier snout. Why is this? Write an alternative linearisation which allows a bounded solution to be obtained.

More generally, an exact characteristic solution of the model allows shocks to form (and thus for the glacier snout to advance). What is the rôle of μ in shock formation.

2. A glacier of depth H is described by the equation

$$H_t + H^{n+1} H_x = a,$$

where the accumulation rate function a varies sinusoidally in time about a mean (space dependent) value; specifically

$$a = a_0(x) + a_1 \sin \omega t,$$

where a_1 is constant.

Linearise the characteristics of the model assuming that H is close to H_0, and determine the resultant form of the perturbed surface. [*It may help to write* $\sin \omega t = \operatorname{Im} e^{i\omega t}$.] What can you say about the effect of millennial scale climate changes? about annual balance changes?

How would you generalise your result to a general time dependent amplitude variation of the form $\displaystyle\int_{-\infty}^{\infty} a_1(\omega) e^{i\omega t}\, d\omega$?

Mathematics and the environment

VACATION SHEET.

1. The drainage pressure in a subglacial channel is determined by the Röthlisberger equations

$$\rho_w g \sin\alpha_c = \frac{f_1 Q^2}{S^{8/3}},$$
$$mL = \rho_w g Q \sin\alpha_c,$$
$$\frac{m}{\rho_i} = KSN^n.$$

Explain the meaning of these equations, and use them to express the effective pressure N in terms of the water flux Q. Find a typical value of N, if $Q \sim 1$ $\mathrm{m^3\ s^{-1}}$, and $\sin\alpha_c \sim 0.1$, $f_1 = f\rho_w g$, $f = 0.05$ $\mathrm{m^{-2/3}\ s^2}$, $n = 3$, $L = 3.3 \times 10^5$ J $\mathrm{kg^{-1}}$ and $K = 0.1$ $\mathrm{bar^{-3}\ y^{-1}}$.

Use a stability argument to explain why Röthlisberger channels can be expected to form an arterial network.

2. The relation between ice volume flux and depth for a surging glacier is found to be a multivalued function, consisting of two monotonically increasing parts, from $(0,0)$ to (H_+, Q_+) and from (H_-, Q_-) to (∞, ∞) in (H, Q) space, where $H_+ > H_-$ and $Q_+ > Q_-$, with a branch which joins (H_-, Q_-) to (H_+, Q_+). Explain how such a flux law can be used to explain glacier surges if the balance function $s(x)$ satisfies $\max s > Q_+$, and give a rough estimate for the surge period.

What happens if $\max s < Q_-$? $\max s \in (Q_-, Q_+)$?

3. The depth of a glacier satisfies the equation

$$H_t + \frac{\partial}{\partial x}\left[(1 - \mu H_x)^n \frac{H^{n+2}}{n+2}\right] = s'(x),$$

where $\mu \ll 1$. Suppose first that $\mu \ll 1$, so that the diffusion term can be neglected. Write down the characteristic solution for an arbitrary initial depth profile. What is the criterion on the initial profile which determines whether shocks will form?

Now suppose $s = 1/(n+2)$ is constant, so that a uniform steady state is possible. Describe the evolution of a perturbation consisting of a uniform increase in depth between $x = 0$ and $x = 1$, and draw the characteristic diagram.

*Shock structure** By allowing $\mu \neq 0$, the *shock structure* is described by the local rescaling $x = x_s(t) + \mu X$. Derive the resulting leading order equation for H, and find a first integral satisfying the boundary conditions $H \to H_\pm$ as $X \to \pm\infty$, where $H_- > H_+$ are the values behind and ahead of the shock. Deduce that the shock speed is

$$\dot{x}_s = \frac{[H^{n+2}]_-^+}{(n+2)[H]_-^+},$$

and that $\phi = H/H_+$ satisfies the equation

$$\phi_\xi = -[g(\phi)^{1/n} - 1],$$

where $\xi = X/H_+$, and

$$g(\phi) = \frac{(r^{n+2} - 1)(\phi - 1) + (r - 1)}{(r - 1)\phi^{n+2}},$$

with $r = H_-/H_+ > 1$. Show that $g(1) = g(r) = 1$, and that $g(\phi) > 1$ for $1 < \phi < r$, and deduce that a monotonic shock structure solution joining H_- to H_+ does indeed exist.

Suppose that $\delta = \Delta H/H_+$ is small, where $\Delta H = H_- - H_+$. By putting $r = 1+\delta$ and $\phi = 1 + \delta\Phi$, show that

$$g = 1 + \frac{\delta^2(n+1)(n+2)}{2}\Phi(1 - \Phi) + \dots,$$

and deduce that

$$\Phi_\Xi \approx -\Phi(1 - \Phi),$$

where

$$\Xi = \frac{\delta(n+1)(n+2)}{2n}\xi.$$

Deduce that the width of the shock structure is of dimensionless order

$$x - x_s \sim \frac{2n\mu H_+}{\delta(n+1)(n+2)},$$

or dimensionally

$$\frac{2n}{(n+1)(n+2)}\frac{d_+^2}{\Delta d \ \tan\alpha},$$

and that for a glacier of depth 100m, slope ($\tan\alpha$) 0.1, with $n = 3$, a wave of height 3 m has a shock structure of width 10 km!