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An Asymptotic Analysis of the Delayed Logistic Equation when the Delay is Large

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We show how to construct an asymptotic solution to the delayed logistic equation $\dot{y} = \alpha y(1-y_1)$, corresponding to the asymptotic limit $\alpha \to \infty$. The results of the analysis are compared with a numerical computation, and found to be comparatively accurate for $\alpha \geqslant 2$. Since the approach adopted is novel, we comment on some features which may be relevant in other problems.

1. Introduction

DIFFERENTIAL DELAY EQUATIONS arise in a variety of practical applications: among these we may mention population growth in ecological systems (Cushing, 1978), the physiology of breathing (Grodins, Buell & Bart, 1967), variation of supply and demand in the economy (Francis *et al.*, 1977), biological immune response (Dibrov, Livshits & Volkenstein, 1977*a*) and various other biochemical cell population models, e.g. of granulocytic leukemia (Mackey & Glass, 1977); delays also appear in realistic models of once-through heat exchangers (Fowler, 1978; Friedly & Krishnan, 1974).

The presence of delay in a system can lead to a notable increase in the complexity of the observed behaviour. Otherwise stable steady states may be destabilized, and as a result, large amplitude oscillations can occur. Even a first order non-linear differential equation can exhibit erratic solutions when a delay is introduced (Mackey & Glass, 1977), and such *chaotic* behaviour is also observed in nature, as well as in discrete difference equations (Li & Yorke, 1975; May, 1976) and non-linear ordinary differential systems (Lorenz, 1963; Rössler, 1977).

Regarding the analysis of solutions of delay-differential equations, bifurcation techniques are applicable (e.g. Chow & Mallet-Paret, 1977) and can be used to construct approximations to small amplitude periodic solutions near the critical value of a relevant bifurcation parameter. Concerning large amplitude oscillations, the position is less good.

If the delay is "small", Taylor series could be used, though there are associated mathematical difficulties, and the delay in any case is not very interesting. For larger delays, Banks (1977) suggests converting a single delay-differential equation to a system of ordinary differential equations, by splitting the delay interval into a large

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number of small intervals, and defining new variables at each point of the interval. Thus if $x(t)$ is the unknown function, put $x_n(t) \equiv x(t-nh)$, and derive additional equations for the x_n by using the Euler approximations $x_n'(t) \approx (x_n - x_{n-1})/h$.

MacDonald (1978) employs a similar method (the linear chain trick) which is applicable to distributed delays of a particular form. For such delays, one obtains (exactly) a finite number of additional linear equations; however, both Banks' and MacDonald's methods are of limited computational use.

When an oscillatory solution is large, expansions in powers of the amplitude are inappropriate, and some other approximation is necessary. In ordinary differential systems, the construction of relaxation oscillations uses the ratio of two (or more) different time scales as a small parameter, and finds the periodic nature of the solution by matching different segments of the solution (Cole, 1968; Stanshine & Howard, 1976). Our aim in this paper is to carry out the equivalent analysis in the case when a delay is present. Since relaxation oscillations usually occur when a small parameter multiplies the highest derivative, it seems that the equivalent case for a delay equation is when the delay is "large". We shall see, however, that the analyses follow quite different tracks.

2. The Delayed Logistic Equation

The logistic equation $dN/dt^* = bN(1-N/K)$ models the growth with time t^* of a population N, which is limited by resources to the attainment of a *saturation* value K: b is the birthrate in the absence of competition. To represent the more realistic case that this competition has a delayed effect on the specific growth rate, Hutchinson (1948) introduced a delay, to obtain the delayed logistic equation

$$dN/dt^* = bN[1-N_\tau/K],$$

where $N_\tau \equiv N(t^*-\tau)$ is the delayed term; the delay τ may represent the maturation time of the individuals in the population. By writing $N = Ky$, $\alpha = b\tau$, $t^* = \tau t$, we may put this in the dimensionless form

$$\frac{dy}{dt} = \alpha y[1-y_1], \quad y_1 \equiv y(t-1); \tag{2.1}$$

α is the ratio of delay to growth time, and the delay is "large" when $\alpha \gg 1$. It will be seen that then $1/\alpha \ll 1$ multiplies the highest derivative, and this, therefore, appears to be the delay equation analogue to a relaxation model.

The literature on (2.1) is substantial. For $0 < \alpha < \pi/2$, $y = 1$ is a stable steady state, but for $\alpha > \pi/2$, it is oscillatorily unstable, and there exists a periodic solution (Jones, 1962*a*, *b*; Hale, 1971) for which $y > 0$. For small $\alpha - \pi/2$, bifurcation techniques (Morris, 1976) show that the period p is approximately given by

$$p \sim 4 + 16\left(\alpha - \frac{\pi}{2}\right)\Big/[\pi(3\pi-2)],$$

and the amplitude A of the corresponding sinusoidal oscillation by

$$A \sim \left[40\left(\alpha - \frac{\pi}{2}\right)\Big/(3\pi-2)\right]^{1/2}$$

Since $A < 1$ for $y > 0$, this cannot be even approximately valid for α larger than

$$\frac{\pi}{2} + (3\pi - 2)/40 \sim 1{\cdot}76;$$

an improved analysis has recently been given by Lin & Kahn (1980), using the transformation $y = \exp u$, also incorporated in the analytic papers by Nussbaum (e.g. 1977, and references therein).

The application of (2.1) to real systems is probably less realistic than it might seem. May (1973) attempted to fit the solution of (2.1) to the Nicholson blowfly experiment (Nicholson, 1954), with limited success. May (1979) and Gurney, Blythe & Nisbet (1980) point out that a delayed recruitment model may be more realistic in considering real populations: nevertheless, the solution of (2.1) is still of interest.

In this paper we consider the asymptotic limit of (2.1) in which $\alpha \to \infty$ and show how to derive explicit analytic expressions for the periodic solution which is known to exist, and which is observed to be numerically stable.

3. Analysis

It has long been known that the stable periodic solution of (2.1) rapidly acquires a spiky form as α increases. To illustrate this, we show in Fig. 1 a numerical solution at $\alpha = 3{\cdot}5$. The solution consists of a series of well-separated pulses. In the flat phase, $y_1 \ll 1$ and the solution must be nearly exponential. Starting with this assumption ($y \sim \exp \alpha t$) and solving by the method of steps (Driver, 1975), quickly leads to the heuristic estimate that the period $p \sim [\exp \alpha]/\alpha$. (Such estimates can, in fact, be proved, but that is immaterial to our present purpose.) This estimate, and the exponential nature of the solution in between pulses, provides the starting point for the analysis which follows.

We suppose that $\exp \alpha t$ is a first approximation (in some sense) to the solution in the flat phase. We focus attention on one pulse, and choose the time origin so that

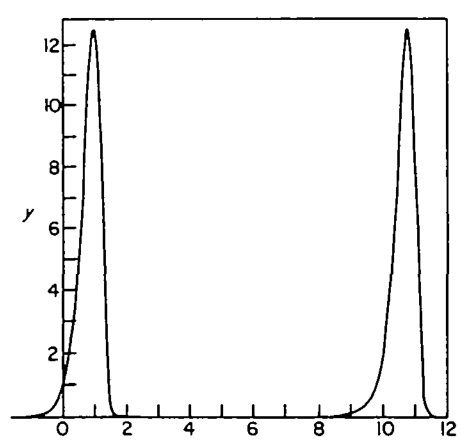

FIG. 1. Numerical solution of (2.1) for $\alpha = 3.5$.

$y(0) = 1$. Thus $y \sim \exp \alpha t$ for $t < 0$. This motivates the introduction of an exponential time scale; we define

$$T = e^{\alpha t}, \quad y(t) \equiv f(T), \quad \delta = e^{-\alpha} \ll 1; \tag{3.1}$$

(2.1) is then

$$Tf'(T) = f(T)[1 - f(\delta T)], \tag{3.2}$$

and we expect $f \simeq T$ for $T < 1$ $(t < 0)$. This suggests the substitution

$$f(T) = Tu(T), \tag{3.3}$$

and u satisfies

$$u'(T) = -\delta u(T)u(\delta T). \tag{3.4}$$

We now commence an asymptotic solution. To do this we make the following observation. We expect the exponential flat phase $f \simeq T$ to be accurate for all $t > -p$, where p is the period of the solution. *Assuming* the heuristic estimate $p \sim (\exp \alpha)/\alpha$ then suggests that $f \sim T$ for $T < 1$ and also

$$T = \exp \alpha t > \exp -\alpha p \sim \exp[-\exp \alpha],$$

i.e. $T > \exp(-1/\delta)$: in other words, the previous pulse occurs at a *transcendentally small* T-scale, which implies that an asymptotic expansion for u in powers of δ will never encounter this pulse. The meaning of this is that we may seek an asymptotic expression for $u(T)$ which is *analytic* at $T = 0$, since the non-analyticity of u (due to the previous pulse) is not relevant to the asymptotic expansion. Having found the solution, we then check *a posteriori* that the period is indeed exponentially large. Thus we seek an asymptotic form of $u(T)$,

$$u(T) \sim U(T), \tag{3.5}$$

where

$$\left.\begin{aligned} U'(T) &= -\delta U(T)U(\delta T), \\ U(\delta T) &= U(0) + \delta T U'(0) + \dots, \end{aligned}\right\} \tag{3.6}$$

and $U(T)$ is assumed analytic at $T = 0$. There is no loss of generality in choosing $U(0) = 1$ (since

$$y(t) = f(T) \sim TU(0) = U(0)e^{\alpha t}$$

for small T, i.e. $t \to -\infty$, and we can always adjust $U(0) = 1$ by choosing the t-origin appropriately). Then (3.6) implies

$$U(0) = 1, \quad U'(0) = -\delta \tag{3.7}$$

etc., so that for small T, the Taylor series for U is just

$$U = 1 - \delta T + O(\delta^2). \tag{3.8}$$

It is easy to see that this is an asymptotic expression for U, provided $T \lesssim 1$. The expansion becomes invalid when $T \sim 1/\delta$, where it is appropriate to write

$$T = \tilde{T}/\delta, \quad U(T) = \tilde{U}(\tilde{T}), \tag{3.9}$$

and $\tilde{U}$ satisfies

$$\tilde{U}'(\tilde{T}) = -\tilde{U}(\tilde{T})\tilde{U}(\delta\tilde{T}), \tag{3.10}$$

with a matching condition (Cole, 1968), that

$$\tilde{U} \sim 1-\tilde{T}+O(\tilde{T}^2)$$

as $\tilde{T} \to 0$ (from (3.8)). Thus asymptotically, for $\tilde{T} \sim 1$, (3.10) may be written

$$\tilde{U}'(\tilde{T}) \sim -\tilde{U}(\tilde{T})[1-\delta\tilde{T}+O(\delta^2)]; \tag{3.11}$$

putting

$$\tilde{U} \sim \tilde{U}_0+\delta\tilde{U}_1+\ldots, \tag{3.12}$$

one easily finds

$$\begin{aligned}\tilde{U}_0 &= Ae^{-\tilde{T}},\\ \tilde{U}_1 &= (B+\tfrac{1}{2}A\tilde{T}^2)e^{-\tilde{T}},\end{aligned} \tag{3.13}$$

and the matching condition that $\tilde{U} \sim 1-\tilde{T}+O(\tilde{T}^2)$ as $\tilde{T} \to 0$ then implies

$$A = 1, \qquad B = 0; \tag{3.14}$$

therefore

$$\tilde{U} \sim e^{-\tilde{T}}[1+\tfrac{1}{2}\delta\tilde{T}^2+O(\delta^2)] = \exp[-\tilde{T}+\tfrac{1}{2}\delta\tilde{T}^2+O(\delta^2)]. \tag{3.15}$$

The asymptotic expression for $\tilde{U}(\delta\tilde{T})$ in (3.11) is no longer valid when $\delta\tilde{T} \sim 1$, and this suggests the choice of a new time scale

$$\tilde{T} = \tilde{\tilde{T}}/\delta, \tag{3.16}$$

and from (3.15), we define

$$\tilde{U} = \exp[-\phi(\tilde{\tilde{T}})/\delta], \tag{3.17}$$

whence ϕ satisfies

$$\phi'(\tilde{\tilde{T}}) = \exp\left[\frac{-\phi(\delta\tilde{\tilde{T}})}{\delta}\right], \tag{3.18}$$

and the matching condition as $\tilde{\tilde{T}} \to 0$ is, from (3.15),

$$\phi \sim \tilde{\tilde{T}}-\tfrac{1}{2}\tilde{\tilde{T}}^2+O(\tilde{\tilde{T}}^3) \quad \text{as } \tilde{\tilde{T}} \to 0. \tag{3.19}$$

From (3.19), it follows that we may replace $\phi(\delta\tilde{\tilde{T}})$ by its asymptotic form in (3.18); we obtain

$$\phi' = \exp[-\tilde{\tilde{T}}+\tfrac{1}{2}\delta\tilde{\tilde{T}}^2+O(\delta^2)] \sim e^{-\tilde{\tilde{T}}}[1+\tfrac{1}{2}\delta\tilde{\tilde{T}}^2+O(\delta^2)], \tag{3.20}$$

and the asymptotic expansion for ϕ is easily found to be

$$\phi \sim a-e^{-\tilde{\tilde{T}}}+\delta[b-\{1+\tilde{\tilde{T}}+\tfrac{1}{2}\tilde{\tilde{T}}^2\}e^{-\tilde{\tilde{T}}}]+O(\delta^2), \tag{3.21}$$

where matching as $\tilde{\tilde{T}} \to 0$ dictates

$$a = b = 1, \tag{3.22}$$

so that

$$\phi \sim 1-e^{-\tilde{\tilde{T}}}+\delta[1-\{1+\tilde{\tilde{T}}+\tfrac{1}{2}\tilde{\tilde{T}}^2\}e^{-\tilde{\tilde{T}}}]+O(\delta^2). \tag{3.23}$$

No further expansions are necessary, since although the asymptotic form of $\phi(\delta\tilde{\tilde{T}})$ used in (3.20) is invalid when $\delta\tilde{\tilde{T}} \sim 1$, nevertheless by this stage ϕ' is transcendentally small, and so further approximations are asymptotically irrelevant. We can now match (3.23) into the entry phase of the next pulse.

Suppose y has period p in t, $y(t-p) = y(t)$. It follows from (3.1) and (3.3) that

$$e^{-ap}u(Te^{-ap}) = u(T). \tag{3.24}$$

We now match $\phi(\bar{\bar{T}})$ into $U(T^+)$, where

$$T^+ = Te^{-\alpha p} = \bar{\bar{T}}e^{-\alpha p}/\delta^2 \tag{3.25}$$

(so $T^+ \approx 1$ when $t \approx p$). We introduce an intermediate variable T_η by putting

$$T^+ = \eta T_\eta, \qquad \bar{\bar{T}} = \delta^2 \eta T_\eta e^{\alpha p}, \qquad \delta^2 e^{\alpha p} \gg 1/\eta \gg 1. \tag{3.26}$$

(We assume that $\delta^2 e^{\alpha p} = \exp[\alpha(p-2)] \gg 1$, i.e. $p > 2$ which is virtually obvious.) Recall that

$$U(T) = \tilde{U}(\tilde{T}) = \exp[-\phi(\bar{\bar{T}})/\delta], \tag{3.27}$$

and from (3.24)

$$U(T) = \exp(-\alpha p)U(T^+); \tag{3.28}$$

it follows from (3.23), (3.26) and (3.27) that

$$U \sim \exp\left\{-\left[\frac{1-e^{-\tilde{T}}}{\delta} + 1 - (1 + \bar{\bar{T}} + \tfrac{1}{2}\bar{\bar{T}}^2)e^{-T} + O(\delta)\right]\right\}$$

$$= \exp\{-[(1/\delta) + 1 + O(\delta) + \mathrm{TST}]\}, \tag{3.29}$$

where TST denotes transcendentally small terms $\{O[\exp(-\delta^2 \eta T_\eta e^{\alpha p})]\}$. But also (3.28), (3.26) and (3.8) show that

$$U \sim \exp(-\alpha p)[1 + O(\eta\delta)]. \tag{3.30}$$

Supposing that we can choose η exponentially small in δ, matching now yields the period p as

$$p \sim (1/\alpha)[(1/\delta) + 1 + O(\delta)]. \tag{3.31}$$

This confirms *a posteriori* the assumption on η.

4. Results

In Section 3 we have given an explicit representation of the periodic solution of (2.1) as $\alpha \to \infty$. We now derive some quantities of interest, and compare them with numerical results.

The period is already given by (3.31). It is

$$p \sim \frac{1}{\alpha}[\exp\alpha + 1 + O(e^{-\alpha})]. \tag{4.1}$$

The maximum value of y, y_{max}, occurs when $t \simeq 1$, i.e. $\tilde{T} \sim 1$, and then $y = \tilde{T}\tilde{u}/\delta$. $y = y_{max}$ when $y' = 0$, i.e. when $\tilde{T}\tilde{u}' + \tilde{u} = 0$, hence from (3.11),

$$1 = \tilde{T}[1 - \delta\tilde{T} + O(\delta^2)],$$

so

$$\tilde{T} = 1 + \delta + O(\delta^2), \tag{4.2}$$

and then using (3.15), we find

$$y_{max} \sim \frac{1}{\delta}e^{-1}[1 + \tfrac{1}{2}\delta + O(\delta^2)],$$

i.e.

$$y_{max} \sim \exp(\alpha-1)+\tfrac{1}{2}e^{-1}+O(e^{-\alpha}). \tag{4.3}$$

The minimum occurs when $\bar{\bar{T}} \sim 1$ $(t \simeq 2)$, when $y \sim (\bar{\bar{T}}/\delta^2)\exp[-\phi(\bar{\bar{T}})/\delta]$. We have $y' = 0$ when $\delta = \bar{\bar{T}}\phi'(\bar{\bar{T}})$, whence (3.20) gives

$$\delta = \bar{\bar{T}}e^{-\bar{\bar{T}}}[1+\tfrac{1}{2}\delta\bar{\bar{T}}^2+O(\delta^2)]. \tag{4.4}$$

This shows $\bar{\bar{T}} \sim \ln(1/\delta)$, and so neglect of the higher order terms gives a less accurate result. Let T_m be the greater root of

$$T_m e^{-T_m} = \delta \tag{4.5}$$

(the smaller is $O(\delta)$ and associated with the maximum). At $\bar{\bar{T}} = T_m$, we have from (3.23)

$$\begin{aligned} y_{min} &\approx y \sim \frac{T_m}{\delta^2}\exp\left\{-\frac{1}{\delta}(1-e^{-T_m})-[1-(1+T_m+\tfrac{1}{2}T_m^2)e^{-T_m}]+O(\delta)\right\}, \\ y_{min} &\approx \frac{T_m}{\delta^2}\exp\left[-\frac{1}{\delta}+\frac{1}{T_m}-1+\delta\left\{\frac{1}{T_m}+1+\tfrac{1}{2}T_m\right\}\right], \end{aligned} \tag{4.6}$$

with an error of $O(\delta)$ in the exponent. A *rough* approximation is, since $T_m \sim \alpha$,

$$y_{min} \approx \alpha\exp(-e^{\alpha}+2\alpha-1). \tag{4.7}$$

It is now clear why the flat phase appears so close to zero. Although the period and the maximum grow exponentially with α, the minimum is *doubly* exponentially small, and so the exponential growth phase has to recover from an exceedingly low value. Even for $\alpha \sim 3$, $y_{min} \sim 10^{-6}$, whereas the period and maximum are both about 7.

In Table 1 we show analytical and numerical results for y_{max}, y_{min} and p; the

TABLE 1

Comparison of numerical solution (upper figures) and analytical solution (lower figures) of the logistic delay equation for various values of α

α	y_{max}	y_{min}	p
1·7	1·943	0·334	4·10
	2·198	0·296	3·808
2·0	2·902	$0{\cdot}685 \times 10^{-1}$	4·40
	2·902	$0{\cdot}794 \times 10^{-1}$	4·195
2·5	4·680	$0{\cdot}160 \times 10^{-2}$	5·36
	4·666	$0{\cdot}181 \times 10^{-2}$	5·273
3·0	7·582	$0{\cdot}173 \times 10^{-5}$	7·07
	7·573	$0{\cdot}188 \times 10^{-5}$	7·029
3·3	10·16	$0{\cdot}290 \times 10^{-8}$	8·55
	10·158	$0{\cdot}312 \times 10^{-8}$	8·519
3·5	12·37	$0{\cdot}111 \times 10^{-10}$	9·77
	12·366	$0{\cdot}177 \times 10^{-10}$	9·747
3·6	13·65	$0{\cdot}422 \times 10^{-12}$	10·46
	13·648	$0{\cdot}444 \times 10^{-12}$	10·444

analytic results are taken from (4.1), (4.3), (4.5) and (4.6), the numerical results from a solution of the equation using a fourth-order Adams–Bashforth method, with step size either 0·01 or 0·005 at higher values of α for numerical stability. The agreement is excellent, even for moderate α.

5. Discussion

Exponentiation of the time ensures that the small parameter is really exp $-\alpha$, not $1/\alpha$, and so the largeness of the delay is enhanced. This is why even moderate delays give very small minima. In the context of population biology, this is very important, since probabilistic effects become relevant when the deterministic population level becomes of the order of one individual. In this way, complete extinction may be effected. Thus a delay in the system is one way of eradicating a population. This idea was used by Dibrov *et al.* (1977*b*) in studying the response of the humoral immune system to an infecting antigen. They also obtained oscillating solutions with flat phases similar to that evident in Fig. 1. In fact, one can extend the analysis of the present paper to their model (Fowler, 1981).

As well as delay equations, other systems display the same slow exponential growth from the zero state, followed by a rapid pulse-like excursion; for example the Lorenz system (Lorenz, 1963) has this behaviour, as well as a postulated physical model for thermal turbulence (Howard, 1966); in the physical sciences, jökulhlaups (Nye, 1976) and spruce budworm populations (Ludwig, Jones & Holling, 1978) also exhibit the same behaviour, and thus the ideas used here may be applicable in a number of other contexts.

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