Homoclinic Bifurcations in $n$ Dimensions

By A. C. Fowler

Bifurcations near homoclinic orbits in $n$ dimensions are described. Depending on the eigenvalues of the Jacobian at the fixed point whose real parts are closest to zero, a strange invariant set of periodic and aperiodic orbits can be produced, which can be described by a Bernoulli shift on a finite set of symbols. These results generalize earlier ones of Shil'nikov, Gaspard, Tresser, Glendinning, and Sparrow, amongst others.

1. Introduction

The existence of chaotic motion in solutions of ordinary differential equations in $\mathbb{R}^n$, $n \geq 3$, implies that strange invariant sets can exist, which contain many aperiodic and periodic orbits. Homoclinic bifurcations are a very important mechanism for causing such sets to come into existence, and their identification and analysis is thus crucial for explaining observations of chaos.

Many such analyses have been carried out since the pioneering work by Shil'nikov [10, 14], who analyzed systems of dimension three and four, respectively. In particular, Shil'nikov himself extended his results to systems of arbitrary dimension to ascertain conditions when a single periodic orbit bifurcates from the homoclinic orbit [11, 13] or to find that, at the bifurcation, a strange invariant set exists [14].

These results are complementary, and were notably advanced by Gaspard [5] and Glendinning and Sparrow [7], who were able to extend Shil'nikov's [10] results to a neighborhood of the parameter value $\mu = 0$ at which the homoclinic orbit exists. A similar extension for the four-dimensional case was done by Fowler and Sparrow [4].

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All these results form part of an overall picture which describes the dynamics of \(n\)-dimensional systems in a parametric neighborhood of a homoclinic bifurcation. However, there are technical difficulties associated with proving the relevant results, some of which have been addressed by Shil'nikov [14], Gaspard [5], and Tresser [16]. Historically, two approaches have been taken. Shil'nikov rewrites the differential equations in integral form and uses smoothness results for the integral equations to derive (and prove) approximations to a Poincaré map for the system. This technique allows analysis in arbitrary finite dimension at the homoclinic bifurcation, but has not been generally applied in a parametric neighborhood of it. A different technique, used by Tresser and coworkers, uses a theorem of Belitski [1] which extends the Hartman-Grobman theorem [9] to linearize exactly (and smoothly) the equations near the origin (the fixed point). The resultant approximate Poincaré map can then be used to study the bifurcation in a parametric neighborhood of the critical value. However, this linearization (essentially via a normal-form procedure) requires that the eigenvalues of the linearized system at the origin be nonresonant, which provides an annoying (and not obviously necessary) restriction on the applicability of the analysis. In particular, it becomes cumbersome for systems of large dimension.

A recent book by Wiggins [17] has shown how the basic analysis of Shil'nikov can be used in \(n\) dimensions. In this paper, we follow the methodology of Wiggins closely, but in addition we show how the derived approximate Poincaré map can be further simplified in order to yield explicit results in the general case.

### 2. An approximate Poincaré map

Consider the system of ordinary differential equations

\[
\dot{x} = f(x, \mu),
\]

where \(\mu \in \mathbb{R}, x \in \mathbb{R}^n\), and \(f\) is a real analytic function. Wiggins [17] has shown that in fact \(f \in \mathcal{C}^2\) is sufficient. We suppose the origin is a fixed point, namely

\[
f(0, \mu) = 0
\]

for all \(\mu\) in a neighborhood of 0, and that when \(\mu = 0\), there exists a homoclinic orbit

\[
\Gamma: x = x^*(t), x^* \to 0 \text{ as } t \to \pm \infty.
\]

It is convenient to specify the phase of \(x^*\) in some way, e.g., by defining \(t = 0\) such that \(|x^*(0)| = \max|\dot{x}^*(t)|\).

We assume that the Jacobian \(Df\) of \(f\) evaluated at \(x = 0, \mu = 0\) has distinct eigenvalues of multiplicity one, none of which has real part equal to zero. This situation then remains true in a neighborhood of \(\mu = 0\). It follows that, near \(\mu = 0\), \(Df\) can be diagonalized over \(\mathbb{C}\) by a linear transformation \(x \to T(\mu)x\).

Furthermore, the stable and unstable manifolds \(W_s\) and \(W_u\) are analytic functions of \(x\), so that, by a local analytic change of variables, we can write \(x = (x_s, x_u)^T, \text{ where } x_s \in W_s, x_u \in W_u, \text{ and (locally) } \mathbb{R}^n = W_s \oplus W_u\). In what follows, we assume both transformations have been carried out, and we write the diagonal matrix \(Df\) as

\[
Df(0, \mu) = D(\mu).
\]

Let \(e^i\) be the normal basis for \(\mathbb{C}^n\). We suppose

\[
x^* \sim \sum_j a_j^s e_j^i \exp(\sigma_j t), \quad t \to -\infty,
\]

\[
x^* \sim \sum_j b_j^u e_j^i \exp(\sigma_j t), \quad t \to +\infty,
\]

where \(D = \text{diag}(\sigma_j)\), and, by our choice of phase for \(x^*\), we may assume \(a_j^s, b_j^u = O(1)\). (For example, if \(x^* = \text{sech} t, \text{ then } (\sigma_j) = (\pm 1)\), and \(a_j^s, b_j^u = 2\).

Letting suffixes \(S\) and \(U\) denote quantities associated with the stable and unstable manifolds of the origin, respectively, we have, with obvious notation,

\[
a^* = (a_{s}^S, 0)^T, \quad b^* = (0, b_{s}^U)^T.
\]

where \(a_{s}^S \in W_s, b_{s}^U \in W_s, (a^*, b^*)\) are the vectors with components \(a_j^s, b_j^u\). Equation (2.5) can be written

\[
x^* \sim \exp(iD_0) a^* \quad \text{as } t \to -\infty,
\]

\[
x^* \sim \exp(iD_0) b^* \quad \text{as } t \to +\infty,
\]

where \(D_0 = D(0) = Df(0, 0)\).

Our purpose now is to obtain an approximate Poincaré map defined on either of the surfaces \(\Sigma\) or \(\Sigma'\), indicated schematically in Figure 1. Here \(\Sigma\) and \(\Sigma'\) are transverse to \(W_s\) and \(W_u\), respectively. Let \(e^U\) and \(e^S\) denote the eigenvectors which correspond to the least unstable and least stable eigenvalue, respectively, i.e. whose real part is closest to zero (when \(\mu \equiv 0\), and hence also for sufficiently small \(\mu \neq 0\)). (In the case of complex conjugates, either will do.) We choose the surfaces \(\Sigma\) and \(\Sigma'\) by the relations

\[
\langle x, e^S \rangle = \nu, \quad \langle x, e^U \rangle = k \nu, \quad k = O(1),
\]

where \(\nu \ll 1\); the precise choice of \(\Sigma'\) is left slightly flexible at this stage for subsequent convenience. Orbits sufficiently close to \(\Gamma\) on \(\Sigma\) will be mapped
under the flow to points in \( \Sigma' \) close to \( \Gamma \). Within the box, \( x = O(\nu) \) and the flow may be approximately linearized. It is convenient to write the system (2.1) in the form

\[
\dot{x} = Dx + g(x),
\]

where

\[
g(x) = f(x, \mu) - Df(0, \mu)x. \tag{2.10}
\]

If \( x = x_0 \) at \( t = t_0 \), the equation (2.9) can be written as the integral equation

\[
x = \exp\left[(t - t_0)D\right]x_0 + \int_{t_0}^{t} \exp\left[(t - t_0 - \tau)D\right]g[x(\tau)] \, d\tau. \tag{2.11}
\]

Since \( g = O(\nu^3) \), standard iterative methods guarantee that \( x \) depends analytically on \( x_0 \) and smoothly on \( t \) for bounded values of \( t \). Further, \( x \) is closely approximated by the linear mapping generated by the linearized flow. (Wiggins [17] shows that the error is \( O(\nu^3) \).)

We choose to define \( \alpha^* \) and \( \beta^* \) in (2.7) precisely by choosing \( t_U \) and \( t_\Sigma \) [both \( O(\ln(1/\nu)) \)] such that

\[
x^* = \exp[-t_U D_0] \alpha^* \quad \text{on} \quad \Sigma', \quad |\langle \alpha^*, e^{\nu} \rangle| = k, \tag{2.12}
\]

\[
x^* = \exp[t_\Sigma D_0] \beta^* \quad \text{on} \quad \Sigma, \quad |\langle \beta^*, e^{\nu} \rangle| = 1;
\]

we are able to do this because \( x^* \) and \( \alpha^* \) on \( \Sigma' \) are both in the local invariant subspace \( W_U \), and similarly for \( x^* \) and \( \beta^* \) on \( \Sigma \). With these choices of \( t_U \) and

\[
t_\Sigma, \text{ we also define} \alpha \text{ and} \beta \text{ by}
\]

\[
x = \exp[-t_U D_0] \alpha \quad \text{on} \quad \Sigma', \quad \left|\langle \alpha, e^{\nu} \rangle\right| = k,
\]

\[
x = \exp[t_\Sigma D_0] \beta \quad \text{on} \quad \Sigma, \quad \left|\langle \beta, e^{\nu} \rangle\right| = 1. \tag{2.13}
\]

If \( \tilde{t} \) is the time of transit from \( \Sigma \) to \( \Sigma' \), then (2.11) can be written in the form

\[
\alpha = \exp[(t_U + t_\Sigma) D_0 + \tilde{t} D] \beta + \exp[t_U D_0] \int_{0}^{\tilde{t}} \exp[(\tilde{t} - \tau) D] g[x(\tau)] \, d\tau. \tag{2.14}
\]

Note that for \( x \) close to \( \Gamma \) in \( \Sigma \) and \( \Sigma' \), we require \( \alpha \) and \( \beta \) to be \( O(1) \).

In order to compute the flow from \( \Sigma' \) back to \( \Sigma \), we suppose that \( x \) is sufficiently close to \( \Gamma \) that a linearization about \( x^* \) can be made. We write

\[
x = x^* + y, \tag{2.15}
\]

so that \( y \) satisfies

\[
y = A_\tau \tau y + \mu \frac{\partial f}{\partial \mu}(x^*, 0) + G(t; y). \tag{2.16}
\]

where

\[
A_\tau = Df(x^*, 0),
\]

\[
G = f(x^* + y, 0) - f(x^*, 0) - Df(x^*, 0)y - \mu \frac{\partial f}{\partial \mu}(x^*, 0). \tag{2.17}
\]

Let \( \Phi(t) \) be a fundamental matrix for the linear equation

\[
\dot{y} = A_\tau y, \tag{2.18}
\]

and define the heteroclinic matrix \( H \) by

\[
\Phi = \exp(iD_0) H(t). \tag{2.19}
\]

By extending Floquet's theorem to orbits of infinite period, we have (see [3, Theorem 8.1]) that \( H \) tends to a constant matrix as \( t \to \pm \infty \). We suppose

\[
H(\infty) H^{-1}(-\infty) = M_0. \tag{2.20}
\]
The solution of (2.16) satisfying \( y = y_0 \) on \( t = t_0 \) can be written
\[
y = \Phi(t)\Phi^{-1}(t_0)y_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s) \left[ G(s;y(s)) + \mu \frac{\partial f}{\partial \mu}(x^*,0) \right] ds. \tag{2.21}
\]

With \( \alpha \) and \( \beta \) defined as in (2.13), we now choose \( k \) such that the time of transit from \( \Sigma' \) to \( \Sigma \) is \( t_U + t_s \). Putting \( t = t_5 \) and \( t_0 = -t_U \), and using (2.19), we obtain
\[
\beta^* - \beta = H(t_5)H^{-1}(-t_U)(\alpha - \alpha^*) + H(t_5)\int_{-t_0}^{t_5} H^{-1}(s)e^{-s^{1/2}}\left[ G + \mu \frac{\partial f}{\partial \mu}(x^*,0) \right] ds. \tag{2.22}
\]

The integral term contains two constituents. That containing \( G \) is (crudely) \( O(y^2) \), and was shown by Wiggins [17] to be \( O(\nu^2) \) (when scaled as here). [This assertion, Proposition (3.2.7) on p. 198, follows from a Taylor expansion of (2.22) about \( \alpha^* \) when \( \mu = 0 \); see equation (3.2.59). There may be some doubt about this, insofar as Wiggins's \( O(\nu^2) \)—our \( O(y^2) \)—will be multiplied by a coefficient which grows as \( \nu \to 0 \) (due to the exponential growth near 0). However, it is nevertheless feasible that the term is indeed \( O(\nu^2) \), since the values of \( x \) in \( \Sigma' \) are restricted so that the image in \( \Sigma \) satisfies \( |x| = O(\nu) \). Nevertheless, the justification for this step is not as straightforward as it may appear.] The second term in \( \alpha \) can be estimated for large \( t_5 \) and \( t_U \), since as \( s \to \pm \infty \),
\[
\frac{\partial f}{\partial \mu}(x^*,0) = D_\mu x^* + O(x^2), \quad \text{where} \quad D_\mu = \frac{dD}{d\mu} \bigg|_{\mu = 0}.
\]

Since \( x^* \sim e^{\nu D_\mu} \), it follows that this term is \( O(\mu [t_5 + t_U + O(1)]) \), where the \( O(1) \) term comes from the nondivergent part of the integral. Lastly, \( H(t_5)H^{-1}(-t_U) = M_0[1 + o(1)] \), where the error terms are \( \exp(-O(t_5, t_U)) \). Since the divergent part of \( \mu \) term arises solely because of the dependence of \( D \) on \( \mu \), it is clearer to include this in the matrix \( M_0 \), so that we write the map (2.22) in the form
\[
\beta^* - \beta = M(\alpha - \alpha^*) + \mu c + O(\nu), \tag{2.23}
\]

where
\[
M = M(\mu) = M_0[1 + O(\mu P)], \tag{2.24}
\]

and where the recurrence time between intersections with \( \Sigma \) is
\[
P = t_U + t_s + \tilde{t}. \tag{2.25}
\]

In a similar way we can write (2.14) in the form
\[
\alpha = \exp(PD_0)[1 + O(\nu)]\beta + O(\nu). \tag{2.26}
\]

The composition of (2.26) with (2.14) defines the Poincaré map.

In what follows, we shall establish results concerning periodic and aperiodic orbits of this map. Wiggins [17] has shown that it suffices to consider \( C^1 \) approximations of this map in doing the analysis. Therefore we henceforth consider the smooth perturbations to the map determined by neglecting the terms of \( O(\mu P, \nu) \) (assuming \( \nu \gg \mu P \)), and drop the suffix zero on \( M_0, D_0 \); the resulting approximation consists of the linear flow
\[
\alpha = e^{P \beta U}, \tag{2.27}
\]

composed with the affine map
\[
\beta^* - \beta = M(\alpha - \alpha^*) + \mu c; \tag{2.28}
\]

this defines a map \( \beta \to \beta^* \) on subsets of \( \Sigma \), with the recurrence time \( P \) being determined via the subsidiary condition which defines \( \Sigma \), which is
\[
|\beta^*| = 1. \tag{2.29}
\]

The general form of this approximate map is not particularly new, although the style of the derivation is rather novel. In particular, by keeping track of the sizes of various terms, we will be able to study the way in which (2.27) and (2.28) approximate the full map, and also see how an approximate one-dimensional map may be further derived from (2.27) and (2.28). The use of further \( C^1 \) approximations to this map may be justified in the same way as before.

3. Geometry of the invariant set

We now wish to approximate (2.27) and (2.28) by using the fact that \( P \) is large for small \( \mu \) and \( \nu \). The idea involved is as follows: a neighborhood of \( \alpha^* \) in \( \Sigma \) of size \( \delta \) will be mapped under (2.28) to a neighborhood of \( \beta^* \) in \( \Sigma \) of size \( \delta' \), provided \( \delta \geq \mu \). The part of this \( N_\delta(\beta^*) \) in \( \Sigma \) which can map through to \( N_{\delta'}(\alpha^*) \) again is close to
\[
\beta_U = e^{-PD_\mu}\alpha_U, \tag{3.1}
\]

where the suffix \( U \) signifies a restriction to \( W_U \). Furthermore, under the map \( \phi \) from \( \Sigma \) to \( \Sigma' \), \( \Sigma \) is mapped into \( \varphi(\Sigma) \), which is close to
\[
\alpha_\Sigma = e^{PD_\mu}\beta_\Sigma. \tag{3.2}
\]
(3.1) denotes a one-dimensional set in \( W_U \), while (3.2) denotes a one-dimensional set in \( W_S \). If \( \dim W_U = k \), \( \dim W_S = n - k \), then \( \dim(W_S \cap \Sigma) = n - k - 1 \), \( \dim(W_U \cap \Sigma) = k - 1 \), and we denote the \((n - k)\)-dimensional set given by (3.1) in \((W_S \cap \Sigma) \cap \Sigma' = \Lambda_0\), and the \(k\)-dimensional set given by (3.2) in \((W_U \cap \Sigma) \cap \Sigma' = \Lambda_0\). The corresponding neighborhoods of \( \Lambda_0 \) and \( \Lambda_0' \) denoted \( \Sigma_0 \) and \( \Sigma_0' \), and we note that our restriction to \( \delta\)-neighborhoods of \( \alpha^* \) and \( \beta^* \) ensures that \( \Sigma_0 \cap W_S \sim \delta \), \( \Sigma_0 \cap W_U \sim \delta \exp(\mathcal{O}(\delta)) \), \( \Sigma_0' \cap W_S \sim \delta \), \( \Sigma_0' \cap W_U \sim \delta \exp(\mathcal{O}(\delta)) \), where the modulus signs denote the size of the indicated set; i.e., \( \Sigma_0 \) is quasi-\((n - k)\)-dimensional, and \( \Sigma_0' \) is quasi-\(k\)-dimensional.

Under the affine transformation (2.28) the quasi-\(k\)-dimensional set \( \Sigma_0' \) is rotated and translated. For \( \mu = 0 \), the point \( \alpha = \alpha^* \) in \( \Sigma' \) is mapped to \( \beta = \beta^* \) in \( \Sigma \). In general, the intersection of the \(k\)-dimensional \( \Lambda_0 \) with the \((n - k)\)-dimensional \( \Lambda_0' \) would have \( n - (n - k) - k \) dimensions, i.e., be a point, but we shall see that when \( \mu = 0 \), the restriction of \( M = M_0 \) to \( W_U \) has rank \( k - 1 \), so that the image \( \varphi(\Lambda_0) \) will be \((k - 1)\)-dimensional. Hence \( \varphi(\Lambda_0) \cap \Lambda_0 = \Lambda_1 \) is one-dimensional, where \( \varphi = \varphi_\mu \) at \( \mu = 0 \), and it follows that \( \varphi(\Sigma_0) \cap \Sigma_1 = \Sigma_1 \) is a neighborhood (of dimension \( n - 1 \)) of a one-dimensional set. The invariant set for the flow lies inside this set, and it is on this basis that a one-dimensional approximation is of some real use.

Beyond our assumption of distinct eigenvalues (so that \( D\phi \) could be diagonalized), we have not assumed any further restriction on the eigenvalues. However, it is evident that if the set of eigenvalues whose real part is closest to zero consists of either two reals, a real and a complex conjugate pair, or two complex conjugate pairs, then a further reduction is possible (and these are the generic possibilities). The latter two cases were considered by Shil'nikov [14] and, more recently, by Glendinning and Sparrow [7], Gaspard [5] and Fowler and Sparrow [4]. The previous statements remain valid, but the intersection of \( \Sigma_0 \) and \( \Sigma_0' \) can be visualized.

We now turn to a justification of the approximations involved in the above discussion. Using suffixes \( U \) and \( S \) to denote elements in \( W_U \) and \( W_S \), we write (2.27) as

\[
\beta_U = e^{-pU}a_U, \\
\alpha_S = e^{pS}b_S. 
\]

(3.3)

Since \( \alpha = \alpha^* \), \( \beta = \beta^* \), it is convenient to define

\[
a_U = a_S^* + a_U, \quad \beta_S = \beta_S^* + b_S, \quad a_U b_S \ll 1, 
\]

(3.4)

and thus

\[
\beta_U = e^{-pU}(a_S^* + a_U), \\
\alpha_S = e^{pS}(\beta_S^* + b_S). 
\]

(3.5)

Notice that, since \( \beta_U \in W_U \), we can choose \( a_U \) in a \((k - 1)\)-dimensional subspace, e.g., by choosing \( a_U^* = 0 \) (this simply defines \( P \)). The Poincaré map \( \Phi = \phi^* \varphi \) from \( \Sigma \) to \( \Sigma \) is defined on the \((n - 1)\)-dimensional hypersurface \( \Sigma \). Equivalently, we map \( b_S \) \((n - k - 1)\)-dimensions, \( a_U \) \((k - 1)\)-dimensions, \( P \) \((1)\) dimension to subsequent values \( b_S, a_U, P \). Thus we define

\[
\beta_U = e^{-pU}(a_S^* + a_U). 
\]

(3.6)

We write

\[
M = \begin{pmatrix} M_{UU} & M_{US} \\ M_{SU} & M_{SS} \end{pmatrix}, 
\]

(3.7)

where \( M_{UU} \) is \( k \times k \), \( M_{US} \) is \( k \times (n - k) \), etc. Thus (2.28) is

\[
b_S = M_{SU}a_U + M_{SS}a_S + \mu c_S, 
\]

(3.8)

and, using (3.6),

\[
e^{-pU}(a_S^* + a_U) = M_{UU}a_U + M_{US}a_S + \mu c_U. 
\]

(3.9)

The idea is that, given \( a_U, P, \) and \( b_S \), (3.5) gives \( a_S \), whereas (3.9) determines \( P' \) and \( a_U \). This is effected by the observation that the rank of \( M_{UU} \) is \( k - 1 \), for the following reason. Since \( \hat{x}^* \) is an exact solution of \( \dot{y} = D\phi(x^*, 0) \), it follows (since \( \hat{x}^* = e^{tD\alpha^*}a_S^* + \mu \hat{x}^* = e^{tD\alpha^*}D\alpha^* \) as \( t \to +\infty \)), that, with relative error \( \alpha(1) \), using (2.21) with \( G = \mu = 0 \),

\[
\beta^* = D_{\alpha^*}^{-1}M_0D_{\alpha^*}. 
\]

(3.10)

Since \( D_0 \) is diagonal and \( \alpha_S^* = 0 = \beta_S^* \), it follows that

\[
M_{UU}(D_{\alpha^*}^{-1}a_S) = 0, 
\]

(3.11)

so that \( \det M_{UU} \neq 0 \). We can assume rank \( M_{UU} = k - 1 \), for otherwise, there exists another \( \alpha_S^* \) such that \( M_{UU}D_{\alpha_S^*} = 0 \), and hence (at \( \mu = 0 \)) another principal homoclinic orbit. This is nongeneric (in the absence of symmetry), and we ignore the possibility.

Suppose \( \eta \) is the (unique) eigenvector of the Hermitian adjoint of \( M_{UU} \); then it follows that we require \( P' \) to satisfy, from (3.9) and (3.5),

\[
\langle \eta, e^{-pU}a_U^* \rangle = \langle \eta, M_{US}e^{pS}\beta_S^* \rangle + \mu \langle \eta, c_U \rangle \\
+ \langle \eta, -e^{pS}a_U + M_{US}e^{pS}b_S \rangle. 
\]

(3.12)

Since the last term is small, then (3.12) is approximately a one-dimensional map
for \( P \) to \( P' \). The Poincaré map \( \Phi : \Sigma \to \Sigma \) is thus defined by (3.12) (to determine \( P' \)), (3.9) (to determine \( a_U \)), and (3.8) (to determine \( b_S \)). All of the various error terms in these equations are analytic functions of \( a_U \), \( b_S \), and \( \mu \), and depend smoothly on \( P \).

4. Periodic and aperiodic orbits

It is convenient to write the Poincaré map given by (3.12), (3.9), and (3.8) in the form

\[
\begin{align*}
\eta, e^{-P \mu} \sigma & = (M_{05} \eta, e^{P \mu} \sigma) + \mu \eta, c_U + \psi(P, P', a_U, b_S), \\
\eta, c_U & = \psi(P, P', a_U, b_S; \mu), \\
b_S & = \psi(a_U, P, P'; \mu),
\end{align*}
\]

(4.1)

where the functions \( \psi \) are analytic in each of their arguments. The equations are defined in a neighborhood of \( a_U = 0 = b_S, P = \infty \), and if this neighborhood is \(|a_U| \leq \delta, |b_S| \leq \delta, |e^{-P \mu}| \leq \delta \), \( |a_U| \) is the magnitude of the real part of the eigenvalue of \( D \) closest to zero, then \( \psi_1 = O(\delta^2), \psi_2 = O(\delta) \) with \( \partial \psi_2 / \partial a_U, \partial \psi_2 / \partial a_U = O(\delta), \partial \psi_2 / \partial b_S = O(\delta) \). Equation (4.1) defines \( a_U \) implicitly, and it is evident that in mapping a ball of size \( \delta \) in \( \Sigma \), the components of \( a_U \) are multiplied by \( O(1/\delta) \), while those of \( b_S \) are multiplied by \( O(\delta) \).

It is clear that the choice of coordinates \((P, a_U, b_S)\) represents a natural decomposition of \( \Sigma \), which we write as \( \Sigma = \Lambda_0 \Phi \Sigma_0 \Sigma_1 \), as above, to parameterize the one-dimensional part of the set, whereas \( \Sigma_0 \Phi \Sigma_1 \) provides a hyperbolic structure for that part of \( \Sigma_1 \), orthogonal to \( \Lambda_1 \), as is illustrated in Figure 2, which represents a transverse section of the set \( \Sigma \), (i.e., it is spanned by \( a_U \) and \( b_S \)). From (4.1), we see that if \( ABCD \) is a \( \delta \)-neighborhood of the origin, then \( \eta \), \( c_U \), \( a_U \) values close to those satisfying \( a_U = \psi(P, P'; 0, \mu) \) are mapped back into \( ABCD \) (where \( a_U = O(\delta) \)). Since \( \Sigma_1 \) is expanding and \( \Sigma_0 \) is contracting, it follows that the image of \( QRS \) in Figure 2 is \( QR' \) as shown (not necessarily with the same orientation), where \( |QT| = O(\delta^2), |QR'| = O(\delta^2) \). Repeating this process in \( \Sigma_1 \Phi \Sigma_0 \) leads to the usual invariant set consisting of a Cantor set (embedded in \( n \)-2 dimensions), if there is more than one value \( a_U = \psi(P, P'; \mu) \) with \( |a_U| \leq \delta \). A much more thorough discussion of the geometry of the Poincaré map is given by Wiggins [17]. As we shall see, this may be the case if either \( \sigma^- \) or \( \sigma^+ \) is complex. It is not difficult to see that the set of values \( a_U = \psi(P, P'; \mu) \) is exactly given [compare (3.12) with (3.9)] by

\[
\eta, e^{-P \mu} \sigma = (M_{05} \eta, e^{P \mu} \sigma) + \mu \eta, c_U,
\]

(4.2)

where \( M_{05}^{-1} \) is the inverse of \( M_{05} \) on the orthogonal complement of \( \eta \) in \( W_U \) and \( P \) is related to \( P \) by the approximate one-dimensional map

\[
\eta, e^{-P \mu} \sigma = (M_{05} \eta, e^{P \mu} \sigma) + \mu \eta, d_U.
\]

(4.3)

From the above discussion, it is straightforward to deduce several features of the bifurcations. Periodic orbits for the flow correspond to fixed points of the map, and are (at least) \( C^1 \)-close to \((P, a_U, b_S)\) satisfying \( P - P \) in (4.3), \( a_U = \psi(P, P'; \mu), b_S = \psi(a_U, P, \mu), \mu \). By the implicit-function theorem, the existence of (non-degenerate) fixed points of (4.3) therefore guarantees the existence of corresponding fixed points for the full Poincaré map. Similarly, period-two orbits, period-four orbits, etc. of (4.3) have corresponding orbits in the flow, and suggest the existence of period-doubling windows. As a consequence, this suggests the possible existence of aperiodic orbits. However, their existence cannot be straightforwardly deduced from the existence of corresponding orbits for the one-dimensional map (4.3), since the use of Bowen's shadowing lemma (see [8]), by which one might prove their existence, relies on some approximated aperiodic sequence \((P, a_U, b_S) \) existing on a hyperbolic invariant set, and while \( \Sigma_0 \Phi \Sigma_1 \) has the appropriate structure, \( \Lambda_1 \) in general will not (because of turning points in the map \( P \to P \) given by (4.3)).

However, we can construct a horshoe for (4.1) if certain conditions are met. Suppose (4.3) is parameterized, so that there are (at least) two values \( a_U = \psi(P, P'; \mu), b_S = \psi(P, P'; \mu) \), say, neighborhoods \( V_U \) and \( V_S \) of which map back to \( |a_U| \leq \delta \). (This occurs if \( \sigma^U \) is complex.) Suppose further that \(|Re(\sigma^U)| > 0 \), so that (4.3) is strongly attractor for *most* values of \( P \). If we can find fixed points \( P_1, P_2 \) of (4.3) on the two components \( V_U \) and \( V_S \), respectively, such that (4.3) contracts the interval \((P_1, P_2)\) in both components, then the Poincaré map induces a hyperbolic structure on \( \Sigma_1 \) neighborhoods of the fixed points corresponding to \( P_1 \) and \( P_2 \).

The situation is illustrated in Figure 3. A vertical strip \( V_1 \) through \( P_1 \) of dimensions \([|a_U|, b_S, e^{-P \mu}] \) of \( O(\delta^2, \delta, \delta) \) is mapped to a set
Figure 3. The vertical sets $V_1$ and $V_2$ containing the fixed points with $P = P_1$ and $P_2$, respectively, are mapped to the horizontal bars $H_1$ and $H_2$. This construction is analogous to that which one obtains in a horseshoe map.

$H_1$, enclosing $P_1$ of dimensions $(δ, δ^2, α(δ))$. A similar set $V_2$ through $P_2$ is mapped to a similar image $H_2$ through $P_2$. The intersections of these four sets, $H_1 \cap V_1 = W_{11}, H_1 \cap V_2 = W_{12}, H_2 \cap V_1 = W_{21}, H_2 \cap V_2 = W_{22}$, when iterated under the Poincaré map $\Phi$, define eight subsets of $W_{ij}$ as

$$W_{11} = \Phi(W_{11}) \cap V_1, \ldots, W_{1k} = \Phi(W_{1k}) \cap V_k, \quad i, j, k = 1, 2,$$

which by construction are nonempty. Iterating $\Phi$ forwards and backwards, we construct (part of) the invariant set for $\Phi$, $W_1 \cup W_2 \cup \ldots$, which is homeomorphic to a symbolic sequence on two symbols. It follows that in this case, the flow contains a countably infinite number of periodic orbits, and uncountably many aperiodic ones. Furthermore, if there are $N$ branches $a_u = \psi(P; \mu)$, with $N$ distinct fixed points, a similar construction leads to a symbolic sequence defined on $N$ symbols. Since the same statements are true for the time-reversed flow, the condition $|\text{Re} \sigma^u_1| > |\text{Re} \sigma^u_2|, \sigma^u \in \mathbb{C}$, may be replaced by $|\text{Re} \sigma^u_1| < |\text{Re} \sigma^u_2|, \sigma^u \in \mathbb{C}$ (Shil'nikov's condition).

Applications

There are three distinct generic cases:

(i) $\sigma^u = \lambda^u, \sigma^s = -\lambda^s$ (Lorenz case),
(ii) $\sigma^u = \lambda^u, \sigma^s = -\lambda^s \pm i\omega^s$ (Shil'nikov case),
(iii) $\sigma^u = \lambda^u \pm i\omega^u, \sigma^s = -\lambda^s \pm i\omega^s$ (bifocal case),

in which $\lambda^{u,s}, \omega^{u,s}$ are real, and no other eigenvalues have the same real parts.

We consider these in turn.

(i) Lorenz Case. Two single real eigenvalues are closest to zero: $\lambda^u > 0$, $-\lambda^s < 0$. If we define

$$\xi = e^{\lambda^u_1 P},$$

then, after suitable rescaling, (4.3) can be written approximately as

$$\xi' = \alpha \xi^\gamma + \mu,$$

where

$$\gamma = \lambda^u / \lambda^u.$$  

This has the form of the Lorenz map for $\xi > 0$ [15], which for the Lorenz equations themselves can be extended antisymmetrically to $\xi < 0$, because of the symmetry of the equations. There is a single fixed point of (4.6) for small $\xi$ and $\mu$, satisfying $\xi = \mu$ for $\gamma > 1$, $\xi = (-\mu / \alpha)^{1/\gamma}$ for $\gamma < 1$, and thus in either event

$$\mu \sim e^{-\lambda_\mu} \xi^\delta,$$

where $\lambda_\mu = \min(\lambda^u, \lambda^u)$, as shown in Figure 4. This was shown by Shil'nikov [11].

(ii) Shil'nikov Case. One single real and one single complex pair of eigenvalues have real parts closest to zero. By appealing to time reversal if necessary, we can assume either that $\lambda^u$ is real and $-\lambda^s \pm i\omega^s$ are complex, or vice versa.

In view of our previous discussion, we take $-\lambda^s$ real, $\lambda^u \pm i\omega^u$ complex. Then (4.3) can be uniformly approximated, possibly after rescaling, by

$$e^{-\lambda^u_1 P} \cos \omega^u_1 P' = \alpha e^{-\lambda^s + \mu},$$

or [using (4.5)]

$$\xi' \cos \left( \Omega \ln \frac{1}{\xi} \right) = \alpha \xi^\gamma + \mu.$$
that when \( \mu = 0 \), the component \( I_s \) can only map into \( I_s \) for those \( s \) satisfying \( s/r \leq \lambda^U/\lambda^U \) (since we require \( ae^{-\mu P} < r < 1 \)), at least for large enough \( P \). This theorem was proved by Shil'nikov [14].

(iii) Bifocal Case. This case, where two single pairs \( \lambda^U \pm i\omega^U, -\lambda^U \pm i\omega^U \) of eigenvalues have real parts closest to zero (one with positive real part, one with negative real part) was also considered by Shil'nikov [14], and the same theorem quoted above applies. Simplification of (4.3) yields the approximate

\[
e^{-\lambda^U P} \cos \Omega P = ae^{-\lambda^U P} \cos (\omega^U P + \phi) + \mu.
\]

(4.15)

Supposing, now without loss of generality, that \( \lambda^U < \lambda^U \), then fixed points are given by

\[
e^{-\lambda^U P} \cos \Omega P = \mu,
\]

(4.16)

and the same considerations as for the Shil'nikov case apply with the same conclusions. The secondary oscillations due to \( \omega^U \neq 0 \) are essentially irrelevant, and (4.13) and (4.14) still apply. When \( \lambda^U = \lambda^U \) in the bifocal case (and in the Shil'nikov case), the situation is more complicated, and has been implicitly considered by Fowler and Sparrow [4], who consider more detail the geometry of the component \( \psi' \) when \( \lambda^U = \lambda^U \), and by Gaspard et al. [5] and Bernhoff [2], who consider the Shil'nikov case when \( \lambda^U = \lambda^U \). The bifocal results also apply if \( \lambda^U > \lambda^U \), and the necessary counterpart of (4.16) is

\[
\mu \sim e^{-\lambda^U P} \cos \omega^U P,
\]

(4.17)

and similarly for (4.14).

**Stability**

Glendinning and Sparrow [7] were able to demonstrate for the three-dimensional Shil'nikov system that tangencies of the one-dimensional map (4.3) [i.e., if we write (4.3) in the form \( L(P') = R(P) \), then \( dL / dP = dR / dP \) and \( P = P' \) correspond to saddle-node bifurcations of the principal periodic orbit in the flow, and that such orbits may be stable. This is due to the fact that in their case \( W_U \) is one-dimensional, so that (4.1) is degenerate, \( a_{11} \) does not exist independently of \( P \), and \( a_{12} = (0) \) in (3.9), which thus defines \( P \) directly. In this case the slope of the map (4.3) \([ = R(P') / L(P') \) helps decide the stability of the orbits. However, if \( \dim W_U \geq 2 \), then all periodic orbits will generally be unstable (typically saddles). In general, we cannot say much about how the periodic orbits disappear near a tangency of (4.3). In cases (ii) and (iii), passage through tangency is associated (when there is a strange invariant set) with disappearance of one of the components \( V_r \). Thus many of the orbits are annihilated. It is tempting to suppose that the principal periodic orbit undergoes a saddle-node bifurcation, and moreover that passage of the slope \( R' / L' \) of (4.3) through \(-1\)
is associated with period doubling. However, while the existence of "period-doubled" orbits can be guaranteed by the implicit-function theorem, the bifurcation sequence cannot be so easily ascertained, since although the Jacobian (and its spectrum) depends continuously on \( \mu \), we cannot be sure that the passage of an eigenvalue through 1 occurs for any particular orbit. Passage through tangency removes many orbits on the invariant set in a complicated way.

5. Conclusions

Homoclinic bifurcations in \( n \)-dimensional flows can cause "explosions," in which strange invariant sets are created. These invariant sets contain countably many periodic orbits and uncountably many aperiodic orbits, all of which are unstable for most values of the bifurcation parameter \( \mu \) (excluding tangencies). If the unstable manifold \( W^u \) of the origin has dimension \( k \), then (most of) the orbits in the invariant set will have \( k-1 \) positive Lyapunov exponents if \( \lambda^u > \lambda^u \), and \( k \) if \( \lambda^s < \lambda^u \), where \( \lambda^s, \lambda^u \) are the absolute values of the real parts of the eigenvalues at the origin which are closest to zero. In general, only three types of bifurcation occur, which are counterparts of their low-dimensional equivalents (Lorenz, Shil’nikov, bifocal examples). The extension to systems with symmetry is obviously an important one here, since such systems will naturally arise in practice (for example, in Fourier truncations of partial differential equations).

Most of our discussion has been descriptive rather than rigorous; a thorough analytic description of the derivation of the Poincaré map, and a description of its geometry, is given by Wiggins [17]. However, Wiggins stops short of analyzing the \( n \)-dimensional case in any detail, referring only to two papers of Shil’nikov [13, 14]. What is novel in this paper, therefore, is the further reduction of the approximate Poincaré map to a perturbation of a one-dimensional map. This further reduction explicitly uses the fact that as \( \mu \to 0 \), the recurrence time \( P \) becomes large. The use of this approximation dramatically simplifies the analysis. In a subsequent paper, we will show how the same formalism can be applied to partial differential equations.

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References