A mathematical model of differential frost heave

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ABSTRACT: The O'Neill-Miller model of frost heave, which takes account of a partially frozen fringe between the frozen and unfrozen soil, is used to study the mechanism of differential frost heave, which is a possible cause of earth hummocks and stone circles. In order to facilitate this study, the model must firstly be generalised to three dimensions, which requires a modification, due to Gilpin, of Miller's concept of regelation; secondly, four key simplifications, variously introduced in previous work by Holden, Fowler and Krantz, must be made to render the computation of the model tractable. With these simplifications, and with the assumption that frozen soil deforms viscously, the model can be reduced to a coupled set of partial differential equations for the frozen soil temperature and velocity. A quasi-stationary stability analysis of the uniform heaving state is conducted on a simplified version of this reduced model to examine whether spatial instabilities can occur in physically realistic conditions. I give an explicit parametric criterion for the occurrence of differential frost heave.

1 INTRODUCTION

Earth hummocks and stone circles are two forms of patterned ground which occur in perennially frozen ground (Tarnocai & Zoltai 1978, Williams & Smith 1989) and which are often thought to occur due to a mechanism of differential frost heave (van Vliet-Lanoë 1991), although other mechanisms have been proposed (e.g. Krantz et al.1988). Differential frost heave refers to a spatial instability which occurs when freezing of the active layer occurs, and which consequently leads to the formation of a pattern consisting of a regular array of hummocks. It is the prediction of this instability which forms the subject of this paper. In order to form a theory of differential frost heave, it is necessary to couple a model of the frost heaving process with a rheological model for the deformation of frozen soil. On the long time scales which are appropriate to frost heave, I consider that a viscous model for the deformation is appropriate. The principal difficulty, however, is that realistic models of frost heaving are notoriously complicated.

The most realistic and most physically based model of frost heave is that due to Miller (1972, 1978). This model was solved numerically by O'Neill & Miller (1982, 1985). The ability to construct a model of differential frost heave relies on the simplifications of the Miller model derived by Fowler & Krantz (1994), following earlier work by Piper et al. (1988). The resulting simplified model can be easily solved for spatially uniform frost heave (Fowler & Noon 1993), and it provides an accurate but accessible model for studying differential frost heave (Noon 1996).

2 THE SIMPLIFIED MILLER MODEL

When soil is frozen at the surface, a freezing front propagates downwards at a rate determined by the surface cooling. At the same time, the capillary effect due to the surface energy of the ice-water interface causes an upwards flow of groundwater, which freezes on to the growing frozen soil, causing frost heave to occur. Commonly this freezing on occurs as a sequence of thin ice lenses, and it is thought that these are formed within a partially frozen fringe, in which ice and water coexist in the pore space. The mathematical model of frost heave solved by O'Neill & Miller (1985) is able to explain in a very satisfactory way this sequence of events. Their model assumes soil to be saturated and incompressible, and it also assumes that ice can move rigidly past soil particles by means of regelation. As explained by Fowler & Krantz (1994), this assumption of rigid ice is untenable in differential frost heave, and it is more realistic to use a model of thermally induced regelation based on work by Romkens & Miller (1973) and Gilpin (1979), in which the ice velocity in the fringe is taken as proportional to temperature gradient. In order to simplify the model, three (accurate) approximations are made, as follows. Firstly, the advection of sensible and latent heat is small; consequently the soil is thermally equilibrated, and its temperature is given by the solution of Laplace's equation. Secondly, the fringe is thin; this is due to the small variation of freezing temperature due to the Gibbs-Thomson effect within the frozen fringe. Typical inferred thicknesses are of the order of millimetres. The implication of this is that the dynamics of the model variables within the fringe may be encapsulated by means of jump conditions across the fringe, so that in the simplified model, one *deduces* that the fringe may be treated as an effective interface. Thirdly, the hydraulic conductivity varies strongly with pore water fraction within the fringe. This allows the pore water pressure profile within the fringe to be solved asymptotically, and allows us to deduce explicit forms for the jump conditions of mass and energy across the fringe.

With these assumptions, one-dimensional frost heave is easily studied, and the problem is easily solved since the temperature profiles are constant. In fact the solution can be given in closed form, as was shown by Fowler & Krantz (1994), and illustrated by Fowler & Noon (1993).

Unlike one-dimensional heave, differential frost heave requires a rheological description of the frozen soil. The rheology of frozen soil has been discussed by Sayles (1988) and Fish (1994); over the long time scales involved here, frozen soil will creep, and for simplicity we assume that it has a constant viscosity η_f.

This viscous flow problem was studied by Noon (1996) and Fowler & Noon (1997). They assumed that the unfrozen soil beneath the frozen soil was rigid by comparison with the frozen soil. At first sight this seems unlikely because surely the unfrozen soil would be softer. However, it is a necessary assumption in the context of the Miller model which assumes rigid (unfrozen) soil in its description. An extension of the model to allow for deformation below the freezing front also requires extension of the Miller model to deal with this, and this is beyond the scope of the present paper. Therefore we follow Fowler & Noon's (1997) assumption. Their study was largely concerned with the effect on stability in the model of a surface snow cover, while here I wish to examine the parameter dependence of regions of instability in the basic model. Fowler & Noon (1997) alluded to the possibility of instability, but their choice of parameters was unphysical (W.B. Krantz, personal communication).

3 STABILITY ANALYSIS

Suppose the vertical coordinate is taken to be z, and that $z_s^0(t)$ and $z_f^0(t)$ denote respectively the soil surface and the freezing front in the one-dimensional solution. We consider three-dimensional perturbations of the one-dimensional heave solutions by writing:

$$z_s = z_s^0 + \zeta, \quad z_f = z_f^0 + \eta \tag{1}$$

and linearising on the basis that $\zeta, \eta << 1$. Selecting

$$\zeta = s(t)\mathrm{e}^{ikx}, \quad \eta = f(t)\mathrm{e}^{ikx} \tag{2}$$

a perturbation of wave number k, thus
and writing the variables in terms of a suitable length scale d and associated heaving time scale

$$\phi h^2 \begin{pmatrix} \dot{s} \\ \dot{f} \end{pmatrix} = M \begin{pmatrix} s \\ f \end{pmatrix} \tag{3}$$

(Fowler & Krantz 1994), we eventually find that:
where $h(t)$ is the (dimensionless) depth of the frozen soil. In deriving Equation (3) we assume (for simplicity) that gravitational effects are negligible and that the unfrozen soil is isothermal. The components of the matrix M are complicated functions which depend on the dimensionless variable $K = kh$, and on two critical parameters: B, the heave parameter, and μ, the differential heave parameter. These are defined by:

$$B = \frac{\beta(\phi - W_l)}{\phi + \beta(\phi - W_l)} \tag{4}$$

and:

$$\mu = 2k^2 \Sigma B' \tag{5}$$

In Equation (4), ϕ is porosity and W_l is the pore water fraction at the top of the frozen fringe; it is a monotone decreasing function of the effective load N; β also depends on load, and is given by:

$$\beta = \frac{\gamma \rho_i L^2}{g\, k_l T_0} \left\{ \frac{K_l (f_l - N)}{-W_l f'_l - \delta\gamma(f_l - N)} \right\} \tag{6}$$

In this equation, γ is the exponent in the permeability dependence on pore water fraction W, ρ_i is ice density, L is latent heat, g is gravity, k_l is the thermal conductivity when $W = W_l$, T_0 is the ambient freezing temperature, δ (≈ 0.1) is the buoyancy ratio between ice and water, K_l is the hydraulic conductivity evaluated at W_l ; f_l is the suction characteristic evaluated at W_l , and f'_l is its derivative with respect to pore water fraction, also evaluated at W_l : further details can be found in Fowler & Krantz (1994). Evidently B is between zero and one, with smaller values for clay, and $O(1)$ values for silts. In Equation (5), B' is the derivative of B with respect to the effective pressure N. It is mostly positive, but negative near saturation (when the effective load is small). The parameter Σ is given by:

$$\Sigma = \frac{\eta_f k_i \Delta T}{\sigma \rho_w d^2 L} \tag{7}$$

where ΔT is the applied surface cooling and σ is a measure of the capillary suction; η_f is the frozen soil viscosity, k_i is the thermal conductivity of frozen soil, ρ_w is the density of water, and d is the depth scale; typical values of Σ are $\geq O(1)$; thus μ is significant, but it can be either positive or negative.

4 DISCUSSION

Equation (3) is a linear evolution equation for the perturbation amplitudes s and f of the top surface and freezing front. The coefficients of the matrix depend on kh, and the parameters B and μ, which thus control whether the perturbations grow. An indication of stability follows from calculating the eigenvalues of the matrix M in Equation (3), and this is tantamount to a frozen time analysis (Robinson 1976). Growth occurs if either eigenvalue has positive real part. Here I simply calculate the eigenvalues to indicate stability or instability. More generally, the presence of the coefficient $\phi h^2 \propto t$ in Equation (3) indicates that unstable modes will grow algebraically in time rather than exponentially, but in addition, the dependence of the eigenvalues on $K = kh$ means that as t and thus h increases, the eigenvalues themselves will change with time.

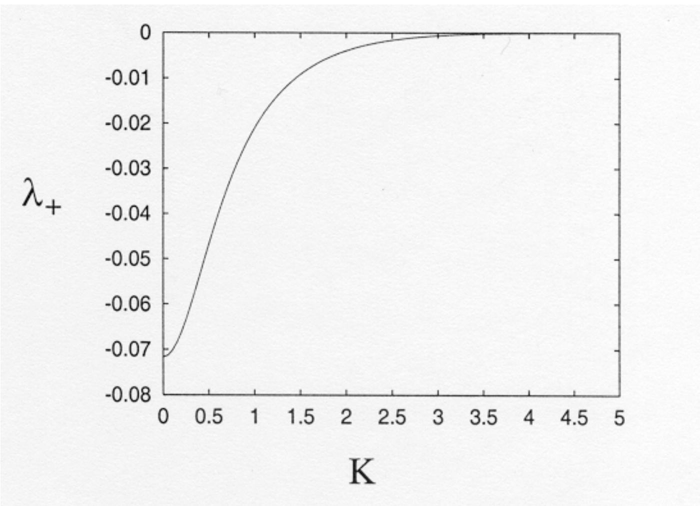

Figure 1. Maximum growth rate as a function of $K = kh$ when $\mu = 0$, $B = 0.1$.

Figure 1 shows a typical plot of the maximum eigenvalue λ_+ of the matrix M in Equation (3) as a function of K, when the heaving parameter B is assumed independent of the effective load N, i.e., $B' = 0$ and thus $\mu = 0$. In this situation both eigenvalues are negative and the heave is stable.

If we allow B to depend on N, however, the situation is different, as was also found by Noon (1996). The principal conclusion is that instability is promoted by positive values of μ. Figure 2 shows a mild instability when $\mu = 2$ and $B = 0.1$, which is greatest at zero wave number, suggesting a long wavelength instability.

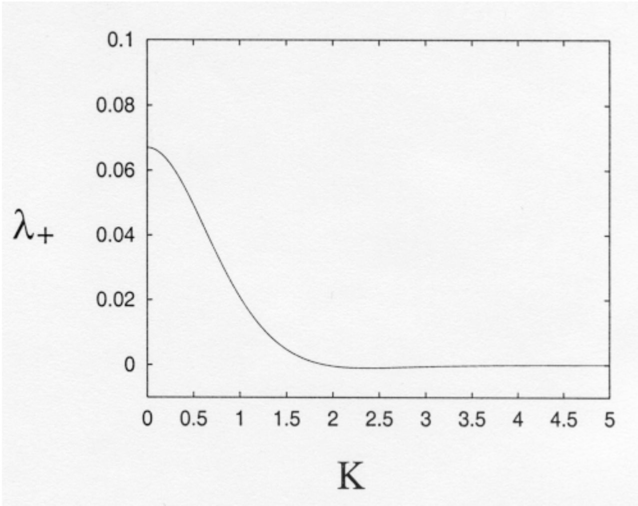

Figure 2. Mild long wavelength instability when $\mu = 2$, $B = 0.1$.

A dramatic feature of the instability is that at slightly higher values of μ, the growth rate increases dramatically at certain wave numbers. This is shown in Figure 3, where there is infinite growth rate at $K \approx 2$. Such rapid growth rate represents a degeneracy in the model. Similar kinds of instability occur in certain biological models (Murray 1993).

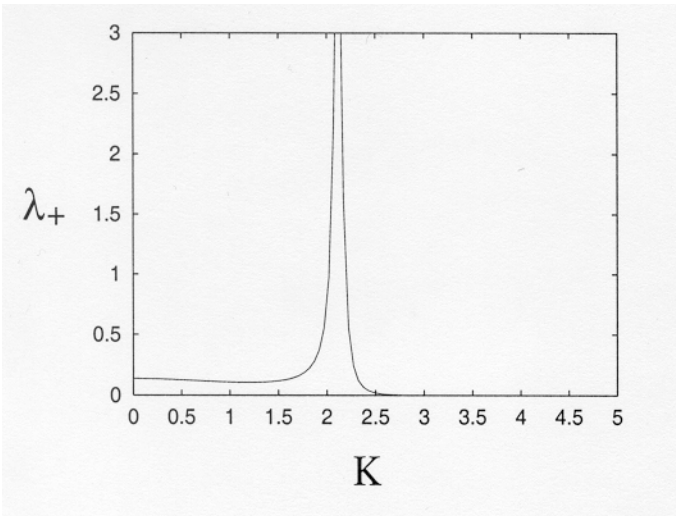

Figure 3. Violent instability when : = 3.111, $B = 0.1$.

Having found instability and consequent differential frost heave when $\mu > 0$, there remains the issue of whether this is physically realistic. The parameter μ is defined in Equation (5), and is given by $\mu = 2k^2\Sigma B'$. Values of Σ can be reasonably large, whereas $0 < B < 1$, with clays having small values.

Insofar as larger values of B are associated with silts, they would appear to be more prone to differential heave. The primary issue is then the sign of μ, since positive values appear to be associated with instability.

The sign of B' depends on the derivative of $\beta(\phi - W_l)$, see Equation (4); β in turn is given by Equation (6). As W_l increases from zero, K_l (which is the hydraulic conductivity in the fringe, and $\propto W_l^\gamma$) (and hence β and also B) increases rapidly, because of the high exponent γ. However, clearly $B \to 0$ as $W_l \to \phi$. Thus B will be a concave function of W_l, with $dB/dW_l < 0$ for values of W_l near saturation. Since W_l is a decreasing function of N, we see that $\mu > 0$ for small N, and thus for small effective pressure, or high pore water pressure. Since heave rate is largest when β is large, which is also when W_l is large, or N is small, it appears that differential frost heave is promoted in conditions under which heave is significant.

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