

## **Preconditioned Iterative Solution of the 2D Helmholtz Equation.**

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Using a finite element method to solve the Helmholtz equation leads to a sparse system of equations which in three dimensions is too large to solve directly. It is also non-Hermitian and highly indefinite and consequently difficult to solve iteratively. The approach taken in this paper is to precondition this linear system with a new preconditioner and then solve it iteratively using a Krylov subspace method. Numerical analysis shows the preconditioner to be effective on a simple 1D test problem, and results are presented showing considerable convergence acceleration for a number of different Krylov methods for more complex problems in 2D, as well as for the more general problem of harmonic disturbances to a non-stagnant steady flow.

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# 1 Introduction

Given an inviscid, irrotational steady flow field described by the potential field  $\phi$  and the associated density  $\rho$  and speed of sound  $c$ , linear unsteady theory considers the addition of a small unsteady perturbation. Considering a single frequency  $\omega$ , the unsteady potential may be expressed as the real part of  $\phi^* \exp(i\omega t)$ , resulting in the linear equation,

$$-\nabla \cdot \rho \nabla \phi^* + \nabla \cdot \left[ \frac{\rho}{c^2} (\nabla \phi \cdot \nabla \phi^* + i\omega \phi^*) \nabla \phi \right] + \frac{\rho}{c^2} (i\omega \nabla \phi \cdot \nabla \phi^* - \omega^2 \phi^*) = 0, \quad (1.1)$$

which will be referred to as the harmonic potential equation. This may be used to model the propagation of acoustic waves emanating from aeroengine inlets.

If the mean flow is stagnant, then  $\nabla \phi = 0$  and (1.1) reduces to the Helmholtz equation

$$-\nabla^2 \phi^* - k^2 \phi^* = 0, \quad (1.2)$$

where  $k = \frac{\omega}{c}$  is the reduced frequency. This equation has applications in many other fields such as electromagnetics and seismology.

The work in this paper is motivated by the desire to perform 3D aeroengine calculations. Adequate resolution of the high reduced frequency waves will require extremely large grids, so direct solution by Gaussian elimination would be prohibitively expensive, even using minimum degree or nested dissection reordering to take maximum advantage of the sparseness of the matrix.

The alternative is to use an iterative solution method, with preconditioning to accelerate the iterative convergence. The problems associated with the iterative solution of the Helmholtz equation are well documented. As  $k$  increases, the discretisation matrix becomes highly indefinite and ill-conditioned. In addition, radiating boundary conditions cause the system to lose its self-adjointness.

There has been considerable work over the last 20 or so years to address these problems. In order to cut down the system's size, the idea of domain decomposition, paving the way for parallel computation, has been used by Fahrat *et al* [9], whilst Chadwick and Bettess found that by expressing the potential in terms of a wave envelope the problem became less oscillatory, thus allowing coarser grids and a smaller resultant system [5]. The fact that the system is complex symmetric was used by Freund [10]. By choosing a certain residual, he halved the work required by the QMR Krylov subspace scheme.

Most work though has been done on preconditioning the discrete system for the use of Krylov subspace iterative solvers. Bayliss *et al* established a benchmark by using a Laplacian preconditioner, approximately inverted with one sweep of SSOR, in conjunction with a CGNR iterative solver [3]. This work was complemented the next year by Gozani *et al* who used multigrid for the approximate inversion of the preconditioner [11]. Baumeister and Kreider added time-dependency into the Helmholtz equation and by setting certain parameters of this dependency, accelerated convergence towards the steady state in what they called a parabolic preconditioner [1]. Two years later, they generalised the idea to produce a so-called hyperbolic preconditioner [2]. Made has recently

attempted to perturb the real part of the matrix to make it positive definite, or at least ‘less indefinite’, before using an incomplete LU preconditioner and a GMRES(m) solver [15]. Otto and Larson used a block preconditioner compatible with Fast Fourier Transform inversion [16], a method also favoured by Elman and O’Leary [8]. Elman has also investigated the use of multigrid on the problem, finding it more efficient to use a GMRES solver on the coarser grids and point Jacobi on the finer [7].

In this paper, we introduce a new elliptic positive definite preconditioner, closely related to the original problem. When used on a simple 1D test problem, numerical analysis shows it to lower the condition number to a level independent of the grid resolution. This is validated by numerical computations and also closely mirrored by a 2D model problem on a unit square with a 1D radiating boundary condition on one side. To avoid the expense of applying the preconditioner exactly, the preconditioner is applied approximately via multigrid in the same manner as Gozani *et al.* [11]. Numerical trials lead to the observation that one single V-cycle with minimal smoothing seems most efficient. This is in agreement with the experiences in [11], and reduces the inversion cost to a level proportional to a residual evaluation.

When coupled with a variety of Krylov subspace schemes, the preconditioner gives considerable convergence acceleration. The favoured schemes are GMRES for the smaller systems, and the CGNR algorithm for the larger ones when GMRES uses too much memory. The attraction of these two schemes is their (theoretical) guaranteed convergence, whereas other methods such as BiCG may, and often do, fail to converge. When a fixed number of points per wavelength is employed on the 2D problem, the number of iterations approaches a linear scaling with the reduced frequency, making the method competitive with other such schemes, whilst at the same time being much more easily generalised to the more complicated harmonic potential equation.

Section 2 introduces the two model problems being investigated. In Section 3 the preconditioner itself is introduced and its efficient approximation discussed in Section 4. The iterative solvers considered are introduced in Section 5. Section 6 looks at some numerical results, comparing both the different iterative methods and the effectivity of the preconditioner. Finally some initial results are presented using the equivalent preconditioner on the linear harmonic potential equation in Section 7, with conclusions being drawn in Section 8.

## 2 The Model Problems

The preconditioner will be first introduced in 1D in a simple problem defined on the unit interval with a source on the left, and a Bayliss-Turkel 1D non-reflecting boundary condition on the right, as shown in Figure 1.

The 2D model problem is defined over the unit square with solid wall boundary conditions on the top and bottom, and once again a source on the left and the 1D non-reflecting boundary conditions on the right. See Figure 2.

For both problems, applying a Galerkin finite element discretisation yields an indef-

$$-\frac{d\phi^*}{dx} = -1 \quad \Big|_{x=0} \quad \frac{d^2\phi^*}{dx^2} - k^2\phi^* = 0 \quad \Big|_{x=1} \quad \frac{d\phi^*}{dx} = -ik\phi^*$$

Figure 1: 1D model problem

$$-\frac{\partial\phi^*}{\partial x} = e^{-\frac{1}{2}(y-\frac{1}{2})^2} \quad \Big|_{x=0} \quad -\nabla^2\phi^* - k^2\phi^* = 0 \quad \Big|_{x=1} \quad \frac{\partial\phi^*}{\partial x} = -ik\phi^*$$

$$\frac{\partial\phi^*}{\partial y} = 0 \quad \Big|_{y=1}$$

$$-\frac{\partial\phi^*}{\partial y} = 0 \quad \Big|_{y=0}$$

Figure 2: 2D model problem

finite system of equations of the form

$$A\phi^* \equiv \left( K - \omega^2 M + B \right) \phi^* = \mathbf{f}, \quad (2.1)$$

where

$K$  is the positive semi-definite, symmetric stiffness matrix,  
 $M$  is the positive definite, symmetric mass matrix,  
 $B$  arises from the radiation boundary condition and is very sparse and non-Hermitian,  
 $\phi^*$  is the vector of unknown nodal approximants to  $\phi$ .

### 3 The Preconditioner

To improve the rate of convergence when solving (2.1) iteratively, it is often preferable to solve the preconditioned equation

$$P^{-1}A\phi^* = P^{-1}\mathbf{f}, \quad (3.1)$$

where the preconditioner  $P$  is chosen so that the condition number of the preconditioned matrix  $P^{-1}A$  is less than that of the original matrix  $A$ .

The new preconditioner introduced in this work is obtained from the Galerkin discretisation by simply omitting  $B$  and changing the sign of the mass matrix term, giving the positive definite symmetric preconditioner

$$P = K + \omega^2 M. \quad (3.2)$$

When using  $N$  points per wavelength for the 1D model problem,

The asymptotic analysis in [14] shows that applying the preconditioner (3.2) to the 1D model problem with  $N$  grid points per wavelength has the following effect on the  $l_2$  condition number:

$$\begin{aligned} \kappa_2(A) &\sim N^2\omega \\ \kappa_2(P^{-1}A) &\sim \omega. \end{aligned} \quad (3.3)$$

A value for  $N$  between 8 and 16 gives reasonable accuracy at an acceptable cost, so the potential benefits of the preconditioning become obvious.

Figure (3) shows the numerical values of  $\kappa_2$  for the 1D and 2D problems. The 1D results agree well with the theory, with  $\kappa_2$  decreasing to a value nearly independent of grid resolution. The 2D results show a similar behaviour, giving confidence in the preconditioner.

Equation form	Algebraic form
Standard equations	$A\phi^* = \mathbf{b}$
Preconditioned standard equations	$P^{-1}A\phi^* = P^{-1}\mathbf{b}$
Normal equations	$A^H A\phi^* = A^H \mathbf{b}$
Preconditioned normal equations	$(P^{-1}A)^H P^{-1}A\phi^* = (P^{-1}A)^H P^{-1}\mathbf{b}$

Table 1: The four possible linear systems to solve, with  $P$  being the preconditioning matrix.

## 4 Multigrid Preconditioning

In practice, one never explicitly constructs the inverse matrix  $P^{-1}$ . Instead, when preconditioning is coupled with a Krylov subspace iterative method, at each iteration there is required at least one solution of a system of equations of the form

$$P \mathbf{p} = \mathbf{d}. \quad (4.1)$$

If this is solved exactly by Gaussian elimination, the computational cost would be as great as the direct solution of the original unpreconditioned equation, and so nothing would have been gained. Instead, the key is to reduce the computational cost of the preconditioner through the use of an approximate solution.

In 1983 Bayliss et al. [3] used the Symmetric Successive Over-Relaxation algorithm to approximately invert a Laplacian preconditioner. In a follow-up paper, Gozani et al. [11] took advantage of the fact that the preconditioner is both symmetric and positive definite to approximately solve (4.1) using multigrid (MG). The advantage of this is that the inversion now has computational complexity per iteration proportional to the number of grid points, see [4]. Using one V-cycle and experimenting with various grid numbers, he found the preconditioner to still be effective but at this vastly reduced cost.

The new preconditioner in this paper is also symmetric positive definite, and so we use the same approach, hopefully retaining most, if not all, of the benefits of the exact preconditioning. To minimise the cost, only one pre- and post-smoothing Jacobi iteration is used within the multigrid cycle, giving a cost which is approximately twice that of evaluating the residual error from the Galerkin equation.

## 5 Krylov Subspace Methods

Krylov Subspace methods, based on the original Conjugate Gradient (CG) algorithm [13] are a large family of iterative schemes for solving sparse matrix systems in one of the four forms shown in Table 1. The choice of which algorithm to use depends on the properties of the system matrix, which in turn depends on the underlying problem itself.

The non-normal equations arising from this problem are neither positive definite nor Hermitian, two properties vital for CG. Hence, a more general technique is required.

$k$	GMRES	GMRES(20)	GMRES(50)	BiCGSTAB(4)	QMR	CGNR
4	10	10	10	2	12	16
8	20	20	20	5	27	38
16	48	227	48	16	80	97
32	207	-	-	289	-	395

Table 2: The number of iterations required for each method to converge for  $N=12.5$ .

The Generalised Minimum Residual method (GMRES) can solve the problem for non-hermitian, indefinite matrices [19]. Like CG, it is robust, guaranteeing convergence in a number of iterations equal to the dimension of the system, but unlike CG the memory requirements increase with the number of iterations. One remedy for this is to periodically restart the algorithm after  $m$  iterations, giving the GMRES( $m$ ) algorithm. However, this results in the loss of the guaranteed convergence, and often the convergence stagnates, giving an incorrect solution.

The other Krylov methods considered here are the BiCGSTAB(4) and QMR algorithms, both popular bi-orthogonal methods but again without any guaranteed convergence properties. Also used is the CGNR algorithm, applying CG iteration to the normal equations. Like full GMRES this method is guaranteed to converge, albeit normally slower than GMRES when that is an option. Unlike GMRES however, a robust bound can be placed on CGNR's convergence rate, linking the number of iterations required to the condition number [6]. Hence, if the condition number can be controlled then so is CGNR's convergence rate.

## 6 Numerical Results

The results in this section are obtained using MATLAB. The solutions are converged to a tolerance of  $10^{-6}$  in the relative residual. If the iteration fails to converge it is indicated by a '-' in the tables.

Table 2 shows the number of iterations for the various methods on the 2D problem when the preconditioner is applied exactly using LU factorisation. As can be seen, both the restarted GMRES and QMR algorithms fail to converge whilst BiCGSTAB(4) seems to also be struggling at the higher  $k$  values. It should be noted here that an iteration of BiCGSTAB(4) is approximately twice as expensive as one of the CGNR algorithm and four times more so than an iteration of GMRES. GMRES is by far the fastest algorithm but its memory problems are also apparent as a larger computer was required for  $k = 32$ . This leaves the CGNR algorithm. Although not the quickest algorithm, its guaranteed and predictable convergence means it is the only reliable algorithm for the larger  $k$  values.

The issue of approximating the conditioner is addressed in Table 3. The preconditioning is implemented approximately via 1 cycle of MG on 4 grids using a relaxed Jacobi smoother with parameter  $\mu = 0.6$ . To minimise the preconditioning cost as much



preconditioning \ $k$	12	24
unconditioned	149	492
1 MG	42	108
LU	38	97

Table 3: The number of iterations required for the CGNR algorithm to converge for  $N=12.5$ , using both exact (LU) and approximate (1 MG) preconditioning.

as possible pre- and post-smoothing was carried out only once on each grid level. Both the approximate and the exact preconditioning are tested in conjunction with the preferred CGNR algorithm. As can be seen, the differences in the number of iterations required for both methods are very small in comparison to the savings made, so most of the theoretical benefits of the preconditioning as discussed in Section 3 are still achieved.

Finally, Figure 4 shows the performance of the CGNR algorithm for three different values of  $N$ , the number of grid points per wavelength. As anticipated, because of the relationship between the convergence rate of CGNR and the condition number of the preconditioned matrix, the graph shows a similar behaviour to the right-hand graph in Figure 3. The closeness of the three preconditioned lines indicates that the convergence rate is almost independent of grid resolution.

## 7 The General Problem

Using similar nomenclature to (2.1), the discrete form of the more general linear harmonic potential equation can be written as

$$\left( K - \omega^2 M + i\omega W + B \right) \phi^* = \mathbf{f}. \quad (7.1)$$

Again  $K$  is real, symmetric, positive semi-definite,  $M$  is real, symmetric, positive definite, and  $B$  is an extremely sparse matrix coming from the radiation boundary conditions.

Following the same approach as for the Helmholtz equation, it seems appropriate to precondition (7.1) with

$$P = K + \omega^2 M. \quad (7.2)$$

Figure 5 shows a domain attempting to model a 2D section of an engine inlet. It is obtained via a conformal mapping of a rectangular domain. Given a mass flow at the fan face (the right-hand boundary of the domain) and freestream conditions on the curved truncated far-field boundary, a FE discretisation of the non-linear Bateman functional [18] yields the steady base flow  $(\rho, \phi, c)$ . (See [12] or [17] for details). This flow is then perturbed, and the general equation (1.1) solved over the domain with an appropriate harmonic source at the fan, and the radiating Bayliss-Turkel condition

$$\frac{\partial \phi^*}{\partial \mathbf{n}} = -ik\phi^* \quad (7.3)$$

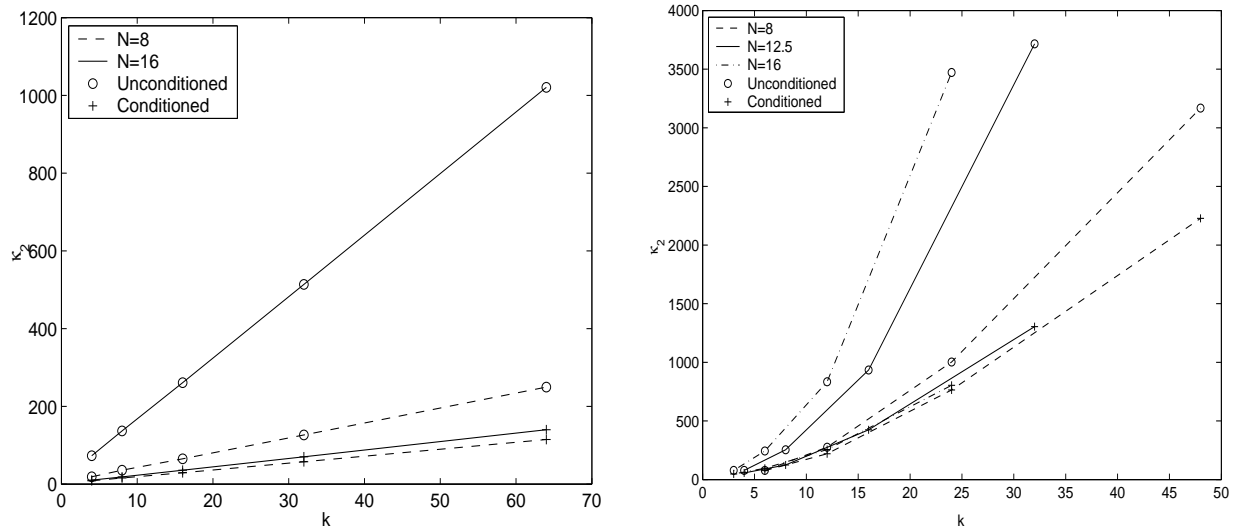


Figure 3: The effects of preconditioning on the condition number in 1D (left) and 2D (right).

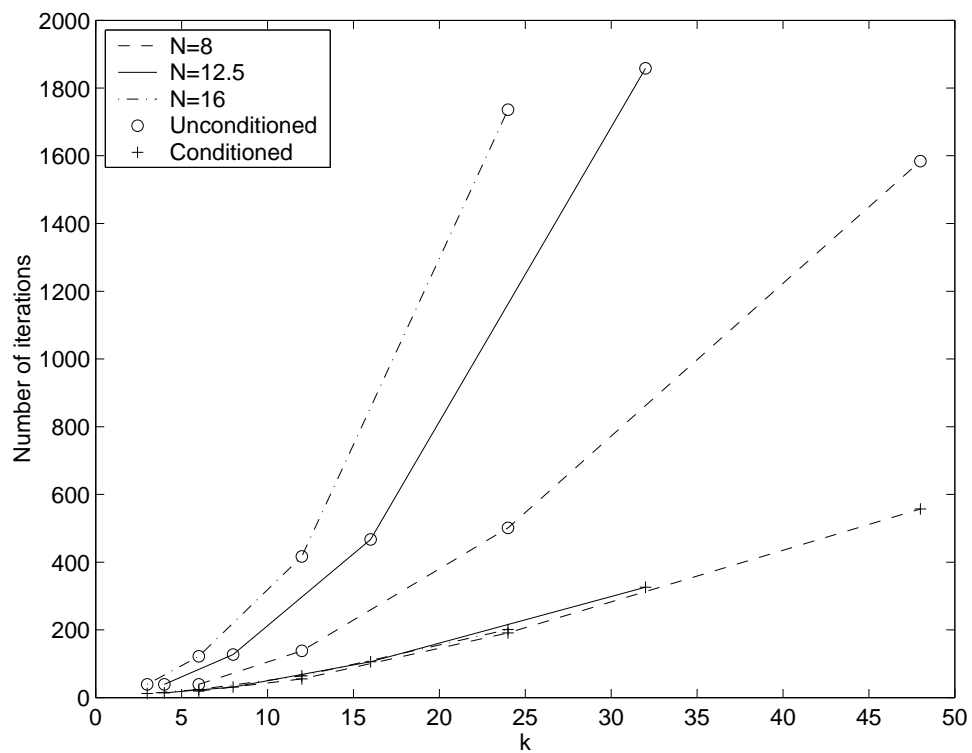


Figure 4: The effects of preconditioning on the convergence of the CGNR algorithm.

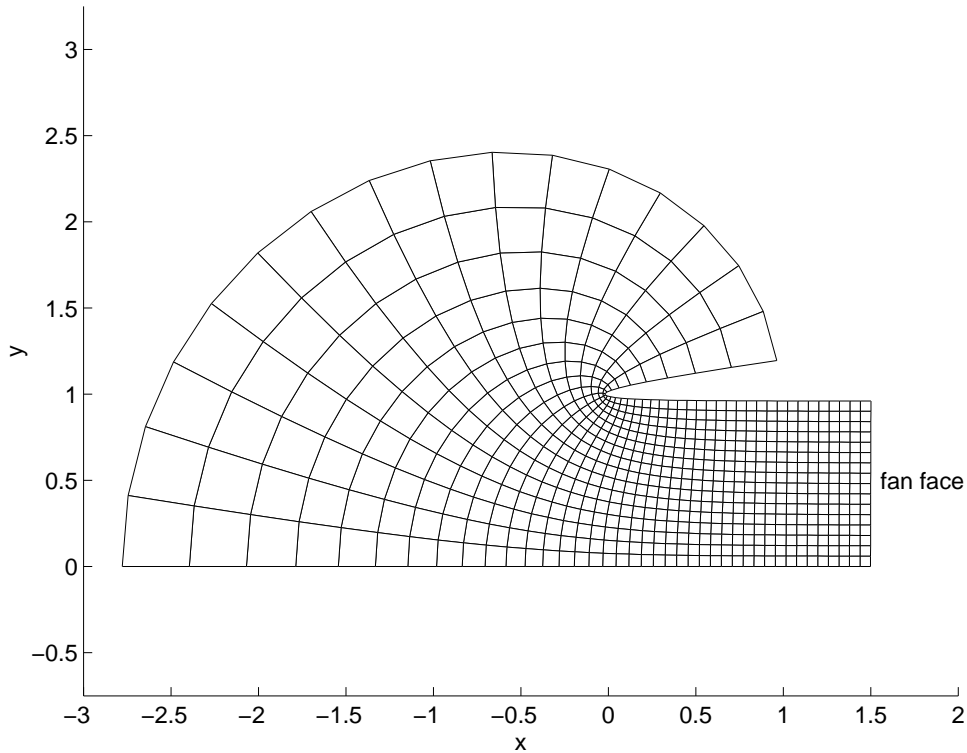


Figure 5: The grid for the general problem.

at the far-field boundary.

The results in Table 4 show the reduction in CGNR iterations when using the preconditioner (7.2) on this problem. The blanks in the three top right boxes are where there are too few grid points per wavelength for a meaningful numerical solution to the problem. The grid size parameter is defined on the rectangular grid, and 1 MG cycle is used to implement the conditioner. Once again, large savings can be seen, especially for the higher  $k$  values on the finer grids. Note also that the convergence dependence on grid resolution is significantly reduced.

## 8 Conclusions

This paper has presented a new and relatively simple way of preconditioning the Helmholtz Equation. Underpinned by sound 1D analysis, when used on a 2D model problem the condition number is drastically reduced to a level almost independent of grid resolution.

Krylov subspace methods are used to solve the discrete Galerkin system, with the GMRES algorithm for small systems and CGNR for larger systems giving the best combination of reliability and computational efficiency. The preconditioner remains effective when implemented approximately, but far more efficiently, through a single multigrid cycle. The method then compares favourably with previous work. Initial

$\Delta x \setminus k$	8	16	32
0.06	268 (729)		
0.03	334 (2288)	803 (2976)	
0.015	433 (8849)	1043 (8163)	2129 (10213)

Table 4: The number of CGNR iterations for the preconditioned (unconditioned) problem with base flow on the domain given in Figure 5.

results for acoustic disturbances on a non-uniform mean flow are very promising, and provide encouragement to proceed to 3D aeroengine computations.

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