

## Stability analysis of numerical interface boundary conditions for parabolic equations

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This paper analyses the numerical stability of coupling procedures in modelling the thermal diffusion in a solid and fluid with continuity of temperature and heat flux at the interface. A simple one-dimensional model is employed with uniform material properties and grid density in each domain. A number of different explicit and implicit algorithms are considered for both the interior equations and the boundary conditions. The analysis shows that, in general, these are stable provided Dirichlet boundary conditions are imposed on the fluid and Neumann boundary conditions are imposed on the solid; in each case, the imposed values are obtained from the other domain.

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# 1 Introduction

This analysis is motivated by interest in numerical procedures for coupling separate computations of thermal diffusion in a solid and a fluid. A typical example application is the computation of heat transfer to a blade in a gas turbine. The surrounding air in a high pressure turbine is on average at a much higher temperature and therefore there is a significant heat flux from the fluid into the turbine blade. In steady-state, this is matched by a corresponding heat transfer from the blade to relatively cold air flowing through internal cooling passages. One approach to the numerical approximation of the above situation would be the use of a single consistent, fully-coupled discretisation modelling both the solid and the fluid, plus the boundary conditions at the interfaces [5]. However, for the solid it is the scalar unsteady parabolic p.d.e. which describes the thermal diffusion, while for the fluid the appropriate equations are the Navier-Stokes equations, with suitable turbulence modelling. Therefore, the production of a single fully-coupled code for the combined diffusion application can be as much work as writing the individual programs for the separate solid and fluid applications. Since there are existing codes which accurately and efficiently solve these individual problems, a more practical approach in many circumstances is to investigate how best to couple such codes together to analyse coupled problems. Both CFD codes and thermal analysis codes usually have the capability to specify either the temperature or the heat flux at boundaries. The natural choice therefore for coupling these codes is to specify the surface temperature at the interface in one code, taking the value from the other code, and specify the boundary heat flux in the second code, taking its value from the first code [1, 2]. A concern is whether there is any possibility that the coupling procedure will introduce a numerical instability which does not exist for the uncoupled problems. This is the issue that is addressed in this study.

The general theory for the analysis of numerical interface or boundary condition instabilities is well-established but can be complicated to apply in practice. In this paper we simplify the analysis for this diffusion problem by restricting attention to a simple 1D model problem, with a uniform grid on either side of the interface. It is usually true that in 2D and 3D computations, any instability due to an interface condition will first occur with a purely 1D eigenmode with a spatial variation in the direction normal to the interface but no variation along the interface. Therefore, the restriction of the analysis to 1D is felt to be well justified. Furthermore, in 2D and 3D flows there is no velocity component normal to the solid boundary of the blade and so in the 1D model problem it is appropriate to omit any convection term.

## 2 Analytic problem

The parabolic p.d.e. describing unsteady thermal diffusion is

$$c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}, \quad q = -k \frac{\partial T}{\partial x} \quad (2.1)$$

Here  $T(x, t)$  is the temperature,  $q(x, t)$  is the heat flux,  $c(x)$  is the heat capacity and  $k(x)$  is the conductivity. These equations are valid for arbitrary, piecewise continuous positive functions  $c(x), k(x)$ . At any interface at which  $c$  and/or  $k$  are discontinuous, the equations are augmented by the requirement that  $T$  and  $q$  must be continuous. The physical requirement that  $T$  is continuous is clear. The requirement that  $q$  is continuous follows from the integral version of the diffusion equation which is

$$\frac{d}{dt} \int_{x_1}^{x_2} c T dx = -[q]_{x_1}^{x_2}. \quad (2.2)$$

If  $x_1$  and  $x_2$  are taken on either side of the interface, then in the limit as  $x_1 \rightarrow x_2$ , both sides of the above equation must approach zero.

Defining

$$T_{max}(t) = \max_x T(x, t), \quad T_{min}(t) = \min_x T(x, t), \quad (2.3)$$

an important property of solutions of the unsteady diffusion equation is that for any non-uniform initial conditions  $T(x, 0)$  and for all  $t > 0$ ,

$$\frac{dT_{max}}{dt} < 0, \quad \frac{dT_{min}}{dt} > 0, \quad (2.4)$$

and  $T_{min}(t) \rightarrow T_{max}(t)$  as  $t \rightarrow \infty$ .

Although the above theory is given for general  $c(x), k(x)$ , in this paper we will now restrict attention to a single interface at  $x = 0$  with  $c$  and  $k$  having uniform values  $c_-, k_-$  for  $x < 0$ , and  $c_+, k_+$  for  $x > 0$ . The boundary conditions as  $x \rightarrow \pm\infty$  are that the temperature approaches a uniform state and so the heat flux tends to zero. The integral version of the thermal diffusion equation will play a central role in the formulation of the discretisations, especially at the interface. Also, the behaviour of the maximum and minimum temperature will be important in defining the numerical stability of the coupled system.

## 3 Fully-coupled discretisation

In this section we examine the stability of fully-coupled discretisations of the model problem. The theory for this is well established since it is simply a special case of the more general problem of the discretisation of a parabolic p.d.e. with spatially varying diffusivity [3, 6]. There are several reasons for doing this analysis even though it is believed that the fully-coupled approach is not the most

practical approach to real coupled applications. The first is to show that in a good fully-coupled discretisation there are no instabilities associated with the interface treatment, that stability of the discretisation on the uniform mesh on either side of the interface is a necessary and sufficient for stability of the fully-coupled discretisation. The second is to have a benchmark against which to compare the ‘weakly-coupled’ discretisations in the next section. These will be shown to have interface instabilities under certain conditions, and it is informative to see how these are related to differences in the interface treatment relative to the fully-coupled discretisation.

Using a computational grid with uniform spacing  $\Delta x_-$  for  $x < 0$  and uniform spacing  $\Delta x_+$  for  $x > 0$ , the location of grid nodes is given by

$$x_j = \begin{cases} j \Delta x_-, & j \leq 0 \\ j \Delta x_+, & j \geq 0 \end{cases} \quad (3.1)$$

Associated with each grid node is the discrete temperature variable  $T_j^n$  which is to approximate the analytic solution  $T(x, t)$  at  $x = x_j$ ,  $t = n\Delta t$ .

### 3.1 An explicit algorithm

Using forward Euler time differencing and conservative spatial differencing based on the integral form of the unsteady diffusion equation on the interval  $x_{j-\frac{1}{2}} \leq x \leq x_{j+\frac{1}{2}}$  gives the following explicit algorithm,

$$C_j (T_j^{n+1} - T_j^n) = -(q_{j+\frac{1}{2}}^n - q_{j-\frac{1}{2}}^n), \quad q_{j+\frac{1}{2}}^n = -K_{j+\frac{1}{2}} (T_{j+1}^n - T_j^n) \quad (3.2)$$

where

$$C_j = \begin{cases} \frac{c_- \Delta x_-}{\Delta t}, & j < 0 \\ \frac{1}{2} \left( \frac{c_- \Delta x_-}{\Delta t} + \frac{c_+ \Delta x_+}{\Delta t} \right), & j = 0 \\ \frac{c_+ \Delta x_+}{\Delta t}, & j > 0 \end{cases} \quad (3.3)$$

and

$$K_{j+\frac{1}{2}} = \begin{cases} \frac{k_-}{\Delta x_-}, & j + \frac{1}{2} < 0 \\ \frac{k_+}{\Delta x_+}, & j + \frac{1}{2} > 0 \end{cases} \quad (3.4)$$

Note that the equation for  $j = 0$  involves the conductivity and heat capacity on both sides of the interface. In particular,  $\Delta t C_0$  is the heat capacity of the whole computational cell extending from  $x_{-\frac{1}{2}}$  to  $x_{+\frac{1}{2}}$ .

For  $j < 0$ , the difference equation reduces to

$$T_j^{n+1} = T_j^n + d_- (T_{j+1}^n - 2T_j^n + T_{j-1}^n) \quad (3.5)$$

where

$$d_- = \frac{k_- \Delta t}{c_- \Delta x_-^2} \quad (3.6)$$

The Fourier mode

$$T_j^n = z^n \exp(ij\theta) \quad (3.7)$$

with the superscript in  $z^n$  being a true exponent not an index denoting the iteration level, is a solution of this difference equation provided that

$$z = 1 - 4d_- \sin^2 \frac{1}{2}\theta. \quad (3.8)$$

Fourier stability requires that  $|z| \leq 1$  for all real values of  $\theta$ ; this is satisfied provided  $d_- \leq \frac{1}{2}$ .

Similarly, Fourier stability of the constant coefficient difference equation in  $j > 0$  requires that  $d_+ \leq \frac{1}{2}$ , where

$$d_+ = \frac{k_+ \Delta t}{c_+ \Delta x_+^2} \quad (3.9)$$

We will now prove that if the requirements of Fourier stability are satisfied on each side of the interface, then the fully-coupled discretisation is stable in the sense that

$$T_{max}^{n+1} \leq T_{max}^n, \quad T_{min}^{n+1} \geq T_{min}^n, \quad (3.10)$$

where

$$T_{max}^n \equiv \max_j T_j^n, \quad T_{min}^n \equiv \min_j T_j^n. \quad (3.11)$$

We begin by noting that if  $d_- \leq \frac{1}{2}$  and  $d_+ \leq \frac{1}{2}$ , then for any positive value  $r$

$$d_- + rd_+ \leq \frac{1}{2}(1+r) \implies \frac{d_- + rd_+}{1+r} \leq \frac{1}{2}. \quad (3.12)$$

In particular, we will use this result with  $r$  as the ratio of the heat capacities of the computational cells on either side of the interface,

$$r = \frac{c_+ \Delta x_+}{c_- \Delta x_-}, \quad (3.13)$$

The next step is to re-write the full difference equation as

$$T_j^{n+1} = (1-a_j-b_j)T_j^n + a_j T_{j+1}^n + b_j T_{j-1}^n \quad (3.14)$$

where

$$\begin{aligned} a_j &= b_j = d_-, & j < 0 \\ a_0 &= \frac{2d_-}{1+r}, & b_0 = \frac{2rd_+}{1+r} \\ a_j &= b_j = d_+, & j > 0 \end{aligned} \quad (3.15)$$

$0 < d_- \leq \frac{1}{2}$  and  $0 < d_+ \leq \frac{1}{2}$ , so for all  $j$ ,  $a_j, b_j$  and  $1 - a_j - b_j$  are positive quantities and thus  $T_j^{n+1}$  is a positive weighted average of  $T_{j+1}^n, T_j^n, T_{j-1}^n$ . Hence,

$$T_{min}^n \leq \min(T_{j+1}^n, T_j^n, T_{j-1}^n) \leq T_j^{n+1} \leq \max(T_{j+1}^n, T_j^n, T_{j-1}^n) \leq T_{max}^n \quad (3.16)$$

This is true for all  $j$ , and so taking the maximum over all  $j$ , and the minimum over all  $j$ , gives the desired result, Eq.(3.10).

### 3.2 An implicit algorithm

Replacing the forward Euler time differencing with backward Euler time differencing gives the following implicit algorithm.

$$C_j (T_j^{n+1} - T_j^n) = -(q_{j+\frac{1}{2}}^{n+1} - q_{j-\frac{1}{2}}^{n+1}) \quad q_{j+\frac{1}{2}}^{n+1} = -K_{j+\frac{1}{2}}(T_{j+1}^{n+1} - T_j^{n+1}), \quad (3.17)$$

with  $C_j, K_{j+\frac{1}{2}}$  as defined before. Fourier stability analysis of the discretisation on either side of the interface yields

$$z = \left(1 + 4d_{\pm} \sin^2 \frac{1}{2}\theta\right)^{-1}. \quad (3.18)$$

In both cases,  $|z| < 1$  for all values of  $d$  and  $\theta$ , giving unconditional Fourier stability for the separate constant coefficient discretisations.

Furthermore, the fully-coupled discretisation is also unconditionally stable in the same sense as before that

$$T_{max}^{n+1} \leq T_{max}^n, \quad T_{min}^{n+1} \geq T_{min}^n. \quad (3.19)$$

To prove this, the difference equation is re-written as

$$T_j^{n+1} = (1 - a_j - b_j) T_j^n + a_j T_{j+1}^{n+1} + b_j T_{j-1}^{n+1} \quad (3.20)$$

where

$$\begin{aligned} a_j &= \frac{K_{j+\frac{1}{2}}}{K_{j+\frac{1}{2}} + K_{j-\frac{1}{2}} + C_j} \\ b_j &= \frac{K_{j-\frac{1}{2}}}{K_{j+\frac{1}{2}} + K_{j-\frac{1}{2}} + C_j} \\ 1 - a_j - b_j &= \frac{C_j}{K_{j+\frac{1}{2}} + K_{j-\frac{1}{2}} + C_j} \end{aligned} \quad (3.21)$$

It is clear that  $a_j, b_j$  and  $1-a_j-b_j$  are positive quantities, for all  $j$ . We now choose  $J$  such that  $T_J^{n+1} = T_{max}^{n+1}$ . Subtracting  $T_{max}^{n+1}$  from both sides of the difference equation gives

$$(1-a_J-b_J)(T_J^n - T_{max}^{n+1}) + a_J(T_{J+1}^{n+1} - T_{max}^{n+1}) + b_J(T_{J-1}^{n+1} - T_{max}^{n+1}) = 0 \quad (3.22)$$

Because  $T_{J-1}^{n+1}, T_{J+1}^{n+1} \leq T_{max}^{n+1}$  and  $a_J, b_J, 1-a_J-b_J$  are all positive, either  $T_J^n > T_{max}^{n+1}$  or  $T_J^n = T_{J+1}^{n+1} = T_{J-1}^{n+1} = T_{max}^{n+1}$ . In the first case, we immediately get the result that  $T_{max}^n > T_{max}^{n+1}$ . In the second case, we can repeat the argument with  $j = J \pm 1$ . By further repetition if necessary, we conclude that either  $T_{max}^n > T_{max}^{n+1}$  or  $T_j^n = T_j^{n+1} = T_{max}^{n+1}$ , for all  $j$ , in which case  $T_{max}^n = T_{max}^{n+1}$ .

Exactly the same argument can be used to prove that  $T_{min}^n \leq T_{min}^{n+1}$  with equality occurring only in the trivial case in which  $T_j^n$  is constant.

## 4 Loosely-coupled discretisation

In the loosely-coupled discretisation, each half of the domain is solved separately with boundary conditions containing information from the other. The natural boundary conditions for a diffusion problem are either Dirichlet (the specification of the boundary temperature) or Neumann (the specification of the boundary heat flux). Therefore we will consider a loosely-coupled procedure in which the calculation for  $x \geq 0$  uses Dirichlet data obtained from the solution for  $x \leq 0$ , while the calculation for  $x \leq 0$  uses Neumann data obtained from the solution for  $x \geq 0$ .

### 4.1 An explicit algorithm

Given existing solutions at time level  $n$  in both halves of the domain, the simplest and most natural explicit numerical algorithm for determining  $T_j^{n+1}$  for  $j \leq 0$  is

$$\begin{aligned} \frac{c_- \Delta x_-}{\Delta t} (T_j^{n+1} - T_j^n) &= \frac{k_-}{\Delta x_-} (T_{j+1}^n - 2T_j^n + T_{j-1}^n), \quad j < 0 \\ \frac{c_- \Delta x_-}{2\Delta t} (T_0^{n+1} - T_0^n) &= -q_w - \frac{k_-}{\Delta x_-} (T_0^n - T_{-1}^n), \end{aligned} \quad (4.1)$$

where  $q_w$  is the heat flux specified as the interface boundary condition. The simplest consistent equation for determining this from the data in  $j \geq 0$  is

$$q_w = -\frac{k_+}{\Delta x_+} (T_1^n - T_0^n). \quad (4.2)$$

The corresponding explicit numerical algorithm for simultaneously determining  $T_j^{n+1}$  for  $j > 0$  is

$$\frac{c_+ \Delta x_+}{\Delta t} (T_j^{n+1} - T_j^n) = \frac{k_+}{\Delta x_+} (T_{j+1}^n - 2T_j^n + T_{j-1}^n). \quad (4.3)$$

The equation for  $j = 1$  requires the variable  $T_0^n$  and this is set by the Dirichlet boundary condition

$$T_0^n = T_w, \quad (4.4)$$

where  $T_w$  is the interface temperature. The obvious value for this is simply  $T_0^n$  from the computation for  $j \leq 0$ .

To summarise the communication between the two calculations for  $j \leq 0$  and  $j \geq 0$ , at each timestep there is an exchange of data, with the program or subroutine performing the calculation for  $j \leq 0$  supplying the value of  $T_w$  to the other program or subroutine performing the calculation for  $j \geq 0$ , while the latter sends  $q_w$  to the former. It is then possible that the computations for the two halves could proceed in parallel (perhaps using separate processes on separate workstations) until they again exchange data before the next timestep.

By comparing Equations (4.1,4.3) with Equation (3.2), it can be seen that the only difference is the omission of the term

$$\frac{c_+ \Delta x_+}{2\Delta t}$$

in the equation for  $j = 0$ . If  $c_+ \Delta x_+ \ll c_- \Delta x_-$ , then this omitted term is negligible compared to the retained term

$$\frac{c_- \Delta x_-}{2\Delta t}$$

and so it seems likely that no instability will be introduced by its omission. On the other hand, if  $c_+ \Delta x_+ \gg c_- \Delta x_-$ , then the omitted term may be very significant. This indicates very simply that a key parameter in the following analysis will be the variable  $r$ , defined earlier in Equation (3.13) as the ratio of these two quantities.

For the purposes of analysis it is more convenient to consolidate and simplify the equations into the following form,

$$\begin{aligned} T_j^{n+1} &= T_j^n + d_- (T_{j+1}^n - 2T_j^n + T_{j-1}^n), & j < 0 \\ T_0^{n+1} &= T_0^n - 2d_- (T_0^n - T_{-1}^n) + 2rd_+ (T_1^n - T_0^n), & (4.5) \\ T_j^{n+1} &= T_j^n + d_+ (T_{j+1}^n - 2T_j^n + T_{j-1}^n), & j > 0 \end{aligned}$$

where  $d_{\pm}$  and  $r$  are as defined previously.

In applying the stability theory of Godunov and Ryabenkii [4, 7], the task is to investigate the existence of separable normal modes of the form

$$T_j^n = z^n f_j. \quad (4.6)$$

The discretisation is unstable if the difference equation admits such solutions which satisfy the far-field boundary conditions,  $|f_j - f_{j-1}| \rightarrow 0$  as  $j \rightarrow \pm\infty$ , and



have  $|z| > 1$ , giving exponential growth in time. The form of the solution is very similar to the assumed Fourier modes, except that the spatial variation is not of the form  $\exp(ij\theta)$ . In fact, for this application the normal mode must be of the form

$$T_j^n = \begin{cases} z^n \kappa_-^j, & j \leq 0 \\ z^n \kappa_+^j, & j \geq 0 \end{cases}. \quad (4.7)$$

The difference equations, Equation (4.5) are satisfied provided the three variables  $z, \kappa_-, \kappa_+$  satisfy the following equations.

$$\begin{aligned} z &= 1 + d_-(\kappa_- - 2 + \kappa_-^{-1}) \\ z &= 1 + 2d_-(\kappa_-^{-1} - 1) + 2rd_+(\kappa_+ - 1) \\ z &= 1 + d_+(\kappa_+ - 2 + \kappa_+^{-1}) \end{aligned} \quad (4.8)$$

Solving the first of these equations to obtain  $\kappa_-^{-1}$  gives

$$\kappa_-^{-1} = 1 - \frac{1-z}{2d_-} \left( 1 \pm \sqrt{1 - \frac{4d_-}{1-z}} \right). \quad (4.9)$$

To satisfy the far-field boundary conditions as  $j \rightarrow -\infty$  it is necessary to choose the negative square root when the argument is real and positive. When it is complex, the choice of root is defined by the requirement that  $|\kappa_-^{-1}| < 1$ .

Similarly, solving the third of the equations gives

$$\kappa_+ = 1 - \frac{1-z}{2d_+} \left( 1 - \sqrt{1 - \frac{4d_+}{1-z}} \right). \quad (4.10)$$

Substituting these into the second equation gives the following nonlinear equation for  $z$ .

$$\sqrt{1 - \frac{4d_-}{1-z}} - r \left( 1 - \sqrt{1 - \frac{4d_+}{1-z}} \right) = 0 \quad (4.11)$$

There is no simple closed form solution to this, giving  $z$  as an explicit function of the parameters  $d_-, d_+, r$ . Instead, we consider asymptotic solutions under different assumptions.

When  $d_-, d_+ \ll 1$ , the square root terms can be expanded to give the following approximate equation and solution.

$$1 - \frac{2d_-}{1-z} - \frac{2rd_+}{1-z} \approx 0, \quad \implies \quad z \approx 1 - 2d_- - 2rd_+. \quad (4.12)$$

The requirement for stability is  $|z| < 1$ . The solution  $z(r)$  lies inside  $|z| = 1$  for sufficiently small values of  $r$ , but then crosses it at  $z = -1$  when  $r = \frac{1}{d_+}$ . Thus for  $d_-, d_+ \ll 1$  the stability requirement is  $0 < r < \frac{1}{d_+}$ .

Expanding the analysis to consider arbitrary values for  $d_-$ ,  $d_+$ , we begin by considering the asymptotic behaviour when  $r \ll 1$  and  $r \gg 1$ .

When  $r \ll 1$  the asymptotic solution is

$$\sqrt{1 - \frac{4d_-}{1-z}} \approx 0, \quad \implies \quad z \approx 1 - 4d_- \quad (4.13)$$

Since  $d_-$  must satisfy  $0 < d_- \leq \frac{1}{2}$  for the discretisation to be stable according to Fourier stability analysis, it follows that  $|z| \leq 1$ . Thus, there is no coupled instability when  $r \ll 1$ .

When  $r \gg 1$ , the asymptotic solution is given by

$$1 - \sqrt{1 - \frac{4d_+}{1-z}} \approx 0, \quad \implies \quad \frac{4d_+}{1-z} \approx 0 \quad \implies \quad |z| \gg 1. \quad (4.14)$$

Given that  $|z| \gg 1$ , an approximate value can then be obtained from

$$1 - \sqrt{1 - \frac{4d_+}{1-z}} \approx -\frac{1}{r} \quad \implies \quad \frac{4d_+}{1-z} \approx \frac{2}{r} \quad \implies \quad z \approx 1 - 2rd_+ \quad (4.15)$$

Thus for fixed  $d_+$  and sufficiently large  $r$ , there is a instability with  $z$  being large, real and negative. The corresponding values of  $\kappa_-^{-1}$  and  $\kappa_+$  will be small, real and negative, so the instability will appear as a ‘sawtooth’ oscillation mode, both spatially and in time, with an amplitude which decays exponentially away from the interface, but grows exponentially in time.

Since the loosely coupled system is stable for  $r \ll 1$  and unstable for  $r \gg 1$  the remaining question is the value of  $r$  at which the instability begins. This corresponds to the lowest positive real value of  $r$  for which  $|z| = 1$ . Because of the requirement that  $r$  is real, it can be shown from Equation (4.11) that this again requires  $z = -1$ , in which case

$$r = \frac{\sqrt{1 - 2d_-}}{1 - \sqrt{1 - 2d_+}} \quad (4.16)$$

Thus the condition for stability is

$$0 < r < \frac{\sqrt{1 - 2d_-}}{1 - \sqrt{1 - 2d_+}} \quad (4.17)$$

A typical calculation using a timestep close to the Fourier stability limit might have  $d_- = d_+ = 0.45$ , for which the coupled stability limit is approximately  $0 < r < 0.46$ . The key to obtaining stability in practical computations is the correct choice of which half of the domain uses the Dirichlet boundary conditions and which half uses the Neumann boundary conditions. The usual practice for the coupled blade/air computations discussed in the Introduction is to use

Neumann boundary conditions for the solid computation, and Dirichlet boundary conditions for the fluid computation. For this choice, the corresponding value of  $r$  is given by

$$r = \frac{c_{\text{fluid}} \Delta x_{\text{fluid}}}{c_{\text{solid}} \Delta x_{\text{solid}}}. \quad (4.18)$$

Given typical values for the parameters involved,  $r$  is usually very small and so this is stable.

If, on the other hand, one were to use Dirichlet boundary conditions for the solid computation and Neumann boundary conditions for the fluid computation, then the appropriate value for  $r$  would be the inverse of the above quantity, which would be very large. In this case the coupled calculation would almost certainly be unstable.

## 4.2 A hybrid algorithm

The next algorithm to consider is a hybrid one, in which the computation is unaltered for  $j > 0$ , but the algorithm for  $j \leq 0$  is replaced by the corresponding implicit method based on a backward Euler time discretisation.

$$\begin{aligned} \frac{c_- \Delta x_-}{\Delta t} (T_j^{n+1} - T_j^n) &= \frac{k_-}{\Delta x_-} (T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}), \quad j < 0 \\ \frac{c_- \Delta x_-}{2\Delta t} (T_0^{n+1} - T_0^n) &= -q_w - \frac{k_-}{\Delta x_-} (T_0^{n+1} - T_{-1}^{n+1}). \end{aligned} \quad (4.19)$$

The boundary heat flux  $q_w$  is again defined explicitly by

$$q_w = -\frac{k_+}{\Delta x_+} (T_1^n - T_0^n). \quad (4.20)$$

The difference equations for  $j > 0$  are unchanged, as is the communication of data between the calculations for  $j \leq 0$  and  $j > 0$ .

The consolidated, simplified form of the equations is

$$\begin{aligned} T_j^{n+1} &= T_j^n + d_- (T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}), \quad j < 0 \\ T_0^{n+1} &= T_0^n - 2d_- (T_0^{n+1} - T_{-1}^{n+1}) + 2rd_+ (T_1^n - T_0^n), \\ T_j^{n+1} &= T_j^n + d_+ (T_{j+1}^n - 2T_j^n + T_{j-1}^n), \quad j > 0 \end{aligned} \quad (4.21)$$

and the normal mode is again of the form

$$T_j^n = \begin{cases} z^n \kappa_-^j, & j \leq 0 \\ z^n \kappa_+^j, & j \geq 0 \end{cases}. \quad (4.22)$$

The difference equations, Equation (4.21) are satisfied provided the three variables  $z, \kappa_-, \kappa_+$  satisfy the following equations.

$$\begin{aligned} 1 &= z^{-1} + d_-(\kappa_- - 2 + \kappa_-^{-1}) \\ 1 &= z^{-1} + 2d_-(\kappa_-^{-1} - 1) + 2rd_+z^{-1}(\kappa_+ - 1) \\ z &= 1 + d_+(\kappa_+ - 2 + \kappa_+^{-1}) \end{aligned} \quad (4.23)$$

The third of these equations requires that  $\kappa_+$  depends on  $z$  in exactly the same way as for the purely explicit algorithm. Solving the first of these equations subject to the far-field boundary conditions gives

$$\kappa_-^{-1} = 1 + \frac{1-z^{-1}}{2d_-} \left( 1 - \sqrt{1 + \frac{4d_-}{1-z^{-1}}} \right). \quad (4.24)$$

Substituting these into the second equation gives

$$\sqrt{1 + \frac{4d_-}{1-z^{-1}}} - r \left( 1 - \sqrt{1 - \frac{4d_+}{1-z}} \right) = 0 \quad (4.25)$$

When  $r \ll 1$  the asymptotic solution is

$$\sqrt{1 + \frac{4d_-}{1-z^{-1}}} \approx 0, \quad \implies \quad z \approx (1 + 4d_-)^{-1} \quad (4.26)$$

$|z| < 1$  for all values of  $d_-$  and so the discretisation is stable.

When  $r \gg 1$ , the asymptotic solution is given by

$$1 - \sqrt{1 - \frac{4d_+}{1-z}} \approx 0, \quad \implies \quad \frac{4d_+}{1-z} \approx 0 \quad \implies \quad |z| \gg 1. \quad (4.27)$$

Given that  $|z| \gg 1$ , an approximate value can then be obtained from

$$\begin{aligned} 1 - \sqrt{1 - \frac{4d_+}{1-z}} &\approx -\frac{\sqrt{1 + 4d_-}}{r} \\ \implies \frac{4d_+}{1-z} &\approx \frac{2\sqrt{1 + 4d_-}}{r} \\ \implies z &\approx 1 - \frac{2rd_+}{\sqrt{1 + 4d_-}} \end{aligned} \quad (4.28)$$

Thus for fixed  $d_-, d_+$  the coupled discretisation is still unstable for sufficiently large values of  $r$ .

The cross-over from stability to instability again occurs when  $z = -1$ , giving

$$r = \frac{\sqrt{1 + 2d_-}}{1 - \sqrt{1 - 2d_+}} \quad (4.29)$$

Thus the condition for stability is

$$0 < r < \frac{\sqrt{1 + 2d_-}}{1 - \sqrt{1 - 2d_+}} \quad (4.30)$$

Comparing this result with the corresponding result for the purely explicit algorithm, it can be seen that the new stability region is greater except when  $d_- \ll 1$ . This has a physical interpretation; when  $d_-$  is not small, the strong implicit coupling of the computational cells for  $j \leq 0$  increases the effective thermal capacity of the cells affected in one timestep by the interface heat flux.

### 4.3 An implicit algorithm

We now consider a fully implicit algorithm, but with explicit updating of the data used for the interface boundary conditions. The implicit numerical algorithm for  $j \leq 0$  is again

$$\begin{aligned} \frac{c_- \Delta x_-}{\Delta t} (T_j^{n+1} - T_j^n) &= \frac{k_-}{\Delta x_-} (T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}), \quad j < 0 \\ \frac{c_- \Delta x_-}{2\Delta t} (T_{0-}^{n+1} - T_{0-}^n) &= -q_w - \frac{k_-}{\Delta x_-} (T_{0-}^{n+1} - T_{-1}^{n+1}). \end{aligned} \quad (4.31)$$

with  $q_w$  defined explicitly by

$$q_w = -\frac{k_+}{\Delta x_+} (T_1^n - T_{0+}^n). \quad (4.32)$$

An important point in the above equations is the distinction between  $T_{0-}^n$ , the value of  $T^n$  at  $j=0$  as calculated for the domain  $j \leq 0$ , and  $T_{0+}^n$ , the value of  $T^n$  at  $j=0$  for the domain  $j \geq 0$ . In the previous discretisations these two values have been identical but this will not be true in this case.

The corresponding implicit numerical algorithm for simultaneously determining  $T_j^{n+1}$  for  $j > 0$  is

$$\frac{c_+ \Delta x_+}{\Delta t} (T_j^{n+1} - T_j^n) = \frac{k_+}{\Delta x_+} (T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}). \quad (4.33)$$

The equation for  $j=1$  requires the variable  $T_{0+}^{n+1}$  and this is set by the Dirichlet boundary condition

$$T_{0+}^{n+1} = T_w, \quad (4.34)$$

where  $T_w$  is the interface temperature. Using explicit updating of boundary data,

$$T_w = T_{0-}^n, \quad (4.35)$$

$T_{0+}$  lags  $T_{0-}$  by one iteration.

The pattern of communication between the calculations for  $j \leq 0$  and  $j \geq 0$  is exactly the same as for the explicit algorithm. They exchange the values of  $T_w$  and  $q_w$  at the beginning of the timestep, perform the timestep calculations independently (possibly in parallel on separate workstations) and then repeat the process for the next timestep.

For the purposes of analysis it is again more convenient to consolidate and simplify the equations into the following form,

$$\begin{aligned} T_j^{n+1} &= T_j^n + d_- (T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}), & j < 0 \\ T_{0-}^{n+1} &= T_{0-}^n - 2d_- (T_{0-}^{n+1} - T_{-1}^{n+1}) + 2rd_+ (T_1^n - T_{0+}^n), & (4.36) \\ T_j^{n+1} &= T_j^n + d_+ (T_{j+1}^{n+1} - 2T_j^{n+1} + T_{j-1}^{n+1}), & j > 0 \\ T_{0+}^{n+1} &= T_{0-}^n. \end{aligned}$$

The form of the normal mode solution for this case is

$$T_j^n = \begin{cases} z^n \kappa_-^j, & j = 0-, -1, -2, -3, \dots \\ z^{n-1} \kappa_+^j, & j = 0+, +1, +2, +3, \dots \end{cases}. \quad (4.37)$$

The fourth equation in Equation (4.36) is automatically satisfied by the above choice of normal mode. The other three equations require that the variables  $z, \kappa_-, \kappa_+$  satisfy the following equations.

$$\begin{aligned} 1 &= z^{-1} + d_- (\kappa_- - 2 + \kappa_-^{-1}) \\ 1 &= z^{-1} + 2d_- (\kappa_-^{-1} - 1) + 2rd_+ z^{-2} (\kappa_+ - 1) & (4.38) \\ 1 &= z^{-1} + d_+ (\kappa_+ - 2 + \kappa_+^{-1}) \end{aligned}$$

Solution of the first and third of these equations, subject to the far-field boundary conditions, gives

$$\begin{aligned} \kappa_-^{-1} &= 1 + \frac{1-z^{-1}}{2d_-} \left( 1 - \sqrt{1 + \frac{4d_-}{1-z^{-1}}} \right), \\ \kappa_+ &= 1 + \frac{1-z^{-1}}{2d_+} \left( 1 - \sqrt{1 + \frac{4d_+}{1-z^{-1}}} \right). \end{aligned} \quad (4.39)$$

Substituting these into the second equation gives

$$\sqrt{1 + \frac{4d_-}{1-z^{-1}}} + rz^{-2} \left( \sqrt{1 + \frac{4d_+}{1-z^{-1}}} - 1 \right) = 0. \quad (4.40)$$

When  $r \ll 1$ , the asymptotic solution is

$$\sqrt{1 + \frac{4d_-}{1-z^{-1}}} \approx 0, \quad \implies \quad z \approx (1 + 4d_-)^{-1} \quad (4.41)$$

$|z| < 1$  for all values of  $d_-$  and so the discretisation is stable.

When  $r \gg 1$ , the asymptotic solution is given by

$$z^{-2} \left( \sqrt{1 + \frac{4d_+}{1-z^{-1}}} - 1 \right) \approx 0, \quad \implies \quad z^{-1} \approx 0 \quad \implies \quad |z| \gg 1 \quad (4.42)$$

Given that  $|z| \gg 1$ , an approximate value can then be obtained from

$$z^{-2} \left( \sqrt{1 + 4d_+} - 1 \right) \approx -\frac{1}{r} \sqrt{1 + 4d_-} \quad \implies \quad z \approx \pm i\sqrt{r} \left( \frac{\sqrt{1 + 4d_+} - 1}{\sqrt{1 + 4d_-}} \right)^{\frac{1}{2}} \quad (4.43)$$

Thus for fixed  $d_-, d_+$  and sufficiently large  $r$ , the coupled system is unstable.

It is not possible for general values of  $d_-, d_+$  to determine explicitly the value of  $r$  above which the solution procedure is unstable. It is possible however to obtain an asymptotic solution under the assumption  $d_-, d_+ \gg 1$ . This is a reasonable assumption since the motivation in using implicit methods is to use much larger timesteps than would be stable using explicit methods. Under the assumption  $d_-, d_+ \gg 1$ , Equation (4.40) reduces to

$$\sqrt{d_-} + rz^{-2}\sqrt{d_+} \approx 0, \quad \implies \quad z \approx \pm i\sqrt{r} \left( \frac{d_+}{d_-} \right)^{\frac{1}{4}}. \quad (4.44)$$

Hence, under these conditions the stability limit is

$$0 < r < \sqrt{\frac{d_-}{d_+}}. \quad (4.45)$$

Provided, as before, that the correct choice is made as to which domain uses the Neumann b.c.'s and which uses the Dirichlet b.c.'s, then  $r$  should be sufficiently small that practical computations will be stable.

## 5 Concluding remarks

The stability analysis in this paper has shown the viability of a loosely-coupled approach to computing the temperature and heat flux in coupled fluid/structure interactions. The key point to achieving numerical stability is the use of Neumann boundary conditions for the structural calculation and Dirichlet boundary conditions for the fluid calculation.

Although the analysis was performed here for the 1D model diffusion equation, the results are believed to be applicable to the real situation in which the 3D diffusion equation is used to model the heat flux in the structure and the 3D Navier-Stokes equations are used to model the behaviour of the fluid. This belief

is supported by the practical experience of 3D computations performed using this coupling procedure [1, 2].

The analysis also assumed a time-accurate modelling of the fluid/structure interaction. In practical computations, the point of engineering interest is often the steady-state temperature and heat flux distributions. In such cases, the computations in the structure and fluid can both proceed with different timesteps given by their respective Fourier stability limits. The coupled normal mode analyses remain valid using the values of  $d_-$ ,  $d_+$  based on the timesteps  $\Delta t_-$ ,  $\Delta t_+$  used in the two domains.

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