Nonreflecting Boundary Conditions for Euler Equation Calculations

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This paper presents a unified theory for the construction of steady-state and unsteady nonreflecting boundary conditions for the Euler equations. These allow calculations to be performed on truncated domains without the generation of spurious nonphysical reflections at the far-field boundaries. The general theory, developed previously by mathematicians, is presented in a more easily understood form based upon fundamental ideas of linear analysis. The application to the Euler equations is given, and the relation to standard "quasi-one-dimensional" boundary conditions is explained. Results for turbomachinery problems show the effectiveness of the new boundary conditions, particularly the steady-state nonreflecting boundary conditions.

I. Introduction

THE objective in formulating nonreflecting boundary conditions is to prevent spurious, nonphysical reflections at inflow and outflow boundaries, so that the calculated flowfield is independent of the location of the far-field boundaries. This leads to greater accuracy and greater computational efficiency, since the computational domain can be made much smaller.

The theoretical basis for nonreflecting boundary conditions stems from a paper by Engquist and Majda,[1] which discusses both ideal nonreflecting boundary conditions and a method for constructing approximate forms, and a paper by Kreiss,[2] which analyzes the wellposedness of initial boundary value problems for hyperbolic systems. Many workers have been active in this area in the last ten years, but their work has been mainly concerned with scalar partial differential equations, with only a couple of recent applications to the Euler equations in specific circumstances.[3,4] Also, almost all of the literature has been written by mathematicians, and in their desire to be absolutely rigorous in their analysis, they use a formalism and assume a background foundation in advanced differential equation theory that makes it difficult for the papers to be appreciated by those with an engineering background.

The author has recently completed a lengthy report on the formulation of nonreflecting boundary conditions and the application to the Euler equations.[5] This report presents a unified view of the theory, with some extensions required for the Euler equations, and does so using the simplest approach possible based upon linear analysis. In taking this approach some rigor is sacrificed, and the conditions for wellposedness become necessary, but possibly not sufficient. The report also shows in full detail the application of the theory to the Euler equations. Another report describes the details of the implementation of the numerical boundary conditions[6] for two-dimensional turbomachinery applications.

The purpose of this paper is to summarize the principal parts of these two reports, and to present results that demonstrate the effectiveness of the new boundary conditions in turbomachinery applications. Because of space limitations, all of the wellposedness analysis, a large amount of algebraic detail, some interesting additional applications, and a variety of helpful comments and insights have been omitted from this paper; the interested reader is urged to refer to the original two reports[5,6] to obtain these.

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II. General Analysis

A. Fourier Analysis

In two dimensions, the analysis is concerned with the following time-dependent, hyperbolic partial differential equation:

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} + B\frac{\partial U}{\partial y} = 0 \tag{1}$$

where U is an N-component column vector and A and B are constant $N \times N$ matrices. We consider wave-like solutions of the form

$$U(x,y,t) = e^{i(kx + ly - \omega t)}\boldsymbol{u}^R \tag{2}$$

where $\boldsymbol{u}^R$ is a constant column vector. Substituting this into the differential equation (1), we find that

$$(-\omega I + kA + lB)\boldsymbol{u}^R = 0 \tag{3}$$

which has nontrivial solutions, provided that

$$\det(-\omega I + kA + lB) = 0 \tag{4}$$

Equation (4) is called the dispersion relation, and it is a polynomial equation of degree N in each of ω, k, and l. We will be concerned with the roots k_n of this equation for given values of ω and l. By dividing the dispersion relation by ω, we obtain

$$\det\left(-I + \frac{k_n}{\omega}A + \frac{l}{\omega}B\right) = 0 \tag{5}$$

and so it is clear that k_n/ω is a function of l/ω. Thus, the variable $\lambda = l/\omega$ will play a key role in constructing all of the boundary conditions.

A critical step in the construction and analysis of boundary conditions is to separate the waves into incoming and outgoing modes. If ω is complex with $\mathrm{Im}(\omega) > 0$ (giving an exponential growth in time), then the right-propagating waves are those for which $\mathrm{Im}(k) > 0$. This is because the amplitude of each wave is proportional to $e^{\mathrm{Im}(\omega)(t - x/c)}$, where $c = \mathrm{Im}(\omega)/\mathrm{Im}(k)$ is the apparent velocity of propagation.

If ω and k are real, then a standard result in the analysis of dispersive wave propagation[7] is that the velocity of energy