Multilevel path simulation for jump-diffusion SDEs

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Abstract We investigate the extension of the multilevel Monte Carlo path simulation method to jump-diffusion SDEs. We consider models with finite rate activity using a jump-adapted discretisation in which the jump times are computed and added to the standard uniform discretisation times. The key component in multilevel analysis is the calculation of an expected payoff difference between a coarse path simulation and a fine path simulation with twice as many timesteps. If the Poisson jump rate is constant, the jump times are the same on both paths and the multilevel extension is relatively straightforward, but the implementation is more complex in the case of state-dependent jump rates for which the jump times naturally differ

1 Introduction

In the Black-Scholes Model, the price of an option is given by the expected value of a payoff depending upon an asset price modelled by a stochastic differential equation driven by Brownian motion,

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$$\mathrm{d}S(t) = a(S(t),t)\,\mathrm{d}t + b(S(t),t)\,\mathrm{d}W(t), \quad 0 \le t \le T, \tag{1}$$

with given initial data S_0. Although this model is widely used, the fact that asset returns are not log-normal has motivated people to suggest models which better capture the characteristics of the asset price dynamics. Merton[Mer76] instead proposed a jump-diffusion process, in which the asset price follows a jump-diffusion SDE:

$$\mathrm{d}S(t) = a(S(t-),t)\,\mathrm{d}t + b(S(t-),t)\,\mathrm{d}W(t) + c(S(t-),t)\,\mathrm{d}J(t), \quad 0 \le t \le T, \tag{2}$$

where the jump term $J(t)$ is a compound Poisson process $\sum_{i=1}^{N(t)}(Y_i - 1)$, the jump magnitude Y_i has a prescribed distribution, and $N(t)$ is a Poisson process with intensity λ, independent of the Brownian motion. Due to the existence of jumps, the process is a càdlàg process, i.e. having right continuity with left limits. We note that $S(t-)$ denotes the left limit of the process while $S(t) = \lim_{s\to t+} S(t)$. In [Mer76], Merton also assumed that $\log Y_i$ has a normal distribution.

There are several ways in which to generalize the Merton model. Here we consider one case investigated by Glasserman & Merener [GM04], in which the jump rate depends on the asset price, namely $\lambda = \lambda(S(t-),t)$.

For European options, we are interested in the expected value of a function of the terminal state, $f(S(T))$, but in the case of exotic options the valuation depends on the entire path $S(t), 0 \le t \le T$. The expected value can be estimated by a simple Monte Carlo method with a suitable approximation to the SDE solution. However, if the discretisation has first order weak convergence then to achieve an $O(\varepsilon)$ root mean square (RMS) error requires $O(\varepsilon^{-2})$ paths, each with $O(\varepsilon^{-1})$ timesteps, leading to a computational complexity of $O(\varepsilon^{-3})$.

Giles [Gil07, Gil08] introduced a multilevel Monte Carlo path simulation method, demonstrating that the computational cost can be reduced to $O(\varepsilon^{-2})$ for SDEs driven by Brownian motion. This has been extended by Dereich and Heidenreich [DH11, Der11] to approximation methods for both finite and infinite activity Lévy-driven SDEs with globally Lipschitz payoffs. The work in this paper differs in considering simpler finite activity jump-diffusion models, but also one example of a more challenging non-Lipschitz payoff, and also uses a more accurate Milstein discretisation to achieve an improved order of convergence for the multilevel correction variance which will be defined later.

We first present the jump-adapted discretisation of jump-diffusion processes, and review the multilevel Monte Carlo method and some modifications for jump-diffusion processes. We then present the numerical algorithm in detail for the constant rate jump-diffusion model, and show numerical results for various options. The next section presents the thinning algorithm used for state-dependent intensities, and the final section draws conclusions and indicates directions for future research.

2 A Jump-adapted Milstein discretisation

To simulate finite activity jump-diffusion processes, we choose to use the jump-adapted approximation proposed by Platen [Pla82]. For each path simulation, the set of jump times $\mathbb{J} = \{\tau_1, \tau_2, \ldots, \tau_m\}$ within the time interval $[0,T]$ is added to a set of uniformly spaced times $t'_i = i\,T/N,\ i = 0,\ldots,N$, to form a combined set of discretisation times $\mathbb{T} = \{0 = t_0 < t_1 < t_2 < \ldots < t_M = T\}$. As a result, the length of each timestep $h_n = t_{n+1} - t_n$ will be no greater than $h = T/N$.

Within each timestep the first order Milstein discretisation is used to approximate the SDE, and then the jump is simulated when the simulation time is equal to one of the jump times. This gives the following numerical method:

$$\begin{aligned}
\widehat{S}^-_{n+1} &= \widehat{S}_n + a_n\, h_n + b_n\, \Delta W_n + \tfrac{1}{2}\, b'_n\, b_n\, (\Delta W_n^2 - h_n), \\
\widehat{S}_{n+1} &= \begin{cases} \widehat{S}^-_{n+1} \; + c(\widehat{S}^-_{n+1}, t_{n+1})(Y_i - 1), & \text{when } t_{n+1} = \tau_i; \\ \widehat{S}^-_{n+1}, & \text{otherwise}, \end{cases}
\end{aligned} \tag{3}$$

where the subscript n is used to denotes the timestep index, $\widehat{S}^-_n = \widehat{S}(t_n-)$ is the left limit of the approximated path, ΔW_n is the Brownian increment during the timestep, a_n, b_n, b'_n are the values of a, b, b' based on $(\widehat{S}_n, t_n)$, and Y_i is the jump magnitude at τ_i.

3 Multilevel Monte Carlo method

For Brownian diffusion SDEs, suppose we perform Monte Carlo path simulations on different levels of resolution ℓ, with 2^ℓ uniform timesteps on level ℓ. For a given Brownian path $W(t)$, let P denote the payoff, and let $\widehat{P}_\ell$ denote its approximation by a numerical scheme with timestep h_ℓ. As a result of the linearity of the expectation operator, we have the following identity:

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]. \tag{4}$$

Let $\widehat{Y}_0$ denote the standard Monte Carlo estimate for $\mathbb{E}[\widehat{P}_0]$ using N_0 paths, and for $\ell > 0$, we use N_ℓ independent paths to estimate $\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ using

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(\widehat{P}^{(i)}_\ell - \widehat{P}^{(i)}_{\ell-1} \right). \tag{5}$$

The multilevel method exploits the fact that $V_\ell := \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ decreases with ℓ, and adaptively chooses N_ℓ to minimise the computational cost to achieve a desired root-mean-square error. This is summarized in the following theorem:

Theorem 1. *Let P denote a functional of the solution of stochastic differential equation (1) for a given Brownian path $W(t)$, and let $\widehat{P}_\ell$ denote the corresponding approximation using a numerical discretisation with timestep $h_\ell = 2^{-\ell}\, T$.*

If there exist independent estimators $\widehat{Y}_\ell$ based on N_ℓ Monte Carlo samples, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that

i) $\left|\mathbb{E}[\widehat{P}_\ell - P]\right| \leq c_1\, h_\ell^\alpha$

ii) $\mathbb{E}[\widehat{Y}_\ell] = \begin{cases} \mathbb{E}[\widehat{P}_0], & l = 0 \\ \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & l > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_\ell] \leq c_2\, N_\ell^{-1} h_\ell^\beta$

iv) C_ℓ, the computational complexity of $\widehat{Y}_\ell$, is bounded by

$$C_\ell \leq c_3\, N_\ell\, h_\ell^{-1},$$

then there exists a positive constant c_4 such that for any $\varepsilon < e^{-1}$ there are values L and N_ℓ for which the multilevel estimator

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell,$$

has a mean-square-error with bound

$$MSE \equiv \mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$$

with a computational complexity C with bound

$$C \leq \begin{cases} c_4\, \varepsilon^{-2}, & \beta > 1, \\ c_4\, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_4\, \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Proof. See [Gil08].

In the case of the jump-adapted discretisation, h_ℓ should be taken to be the uniform timestep at level ℓ, to which the jump times are added to form the set of discretisation times. We have to define the computational complexity as the expected computational cost since different paths may have different numbers of jumps. However, the expected number of jumps is finite and therefore the cost bound in assumption *iv*) will still remain valid for an appropriate choice of the constant c_3.

4 Multilevel Monte Carlo for constant jump rate

The Multilevel Monte Carlo approach for a constant jump rate is straightforward. The jump times τ_j, which are the same for the coarse and fine paths, are simulated by setting $\tau_j - \tau_{j-1} \sim \exp(\lambda)$. The Brownian increments ΔW_n are generated for the fine path, and then summed appropriately to generate the increments for the coarse path. In the following we show numerical results for European call, lookback and barrier options. Asian and digital options have also been simulated; numerical results for these are available in [Xia11] along with more details of the construction of the multilevel estimators for the path-dependent payoffs.

All of the options are priced for the Merton model in which the jump-diffusion SDE under the risk-neutral measure is

$$\frac{\mathrm{d}S(t)}{S(t-)} = (r - \lambda m)\,\mathrm{d}t + \sigma\,\mathrm{d}W(t) + \mathrm{d}J(t), \quad 0 \le t \le T,$$

where λ is the jump intensity, r is the risk-free interest rate, σ is the volatility, the jump magnitude satisfies $\log Y_i \sim N(a,b)$, and $m = \mathbb{E}[Y_i] - 1$ is the compensator to ensure the discounted asset price is a martingale. All of the simulations in this section use the parameter values $S_0 = 100$, $K = 100$, $T = 1$, $r = 0.05$, $\sigma = 0.2$, $a = 0.1$, $b = 0.2$, $\lambda = 1$.

4.1 European call option

Figure 1 shows the numerical results for the European call option with payoff $\exp(-rT)\,(S(T)-K)^+$, with $(x)^+ \equiv \max(x,0)$ and strike $K = 100$.

The top left plot shows the behaviour of the variance of both $\widehat{P}_\ell$ and the multilevel correction $\widehat{P}_\ell - \widehat{P}_{\ell-1}$, estimated using 10^5 samples so that the Monte Carlo sampling error is negligible. The slope of the MLMC line indicates that $V_\ell \equiv \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}] = O(h_\ell^2)$, corresponding to $\beta = 2$ in condition *iii*) of Theorem 1. The top right plot shows that $\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ is approximately $O(h_\ell)$, corresponding to $\alpha = 1$ in condition *i*). Noting that the payoff is Lipschitz, both of these are consistent with the first order strong convergence proved in [Pla10].

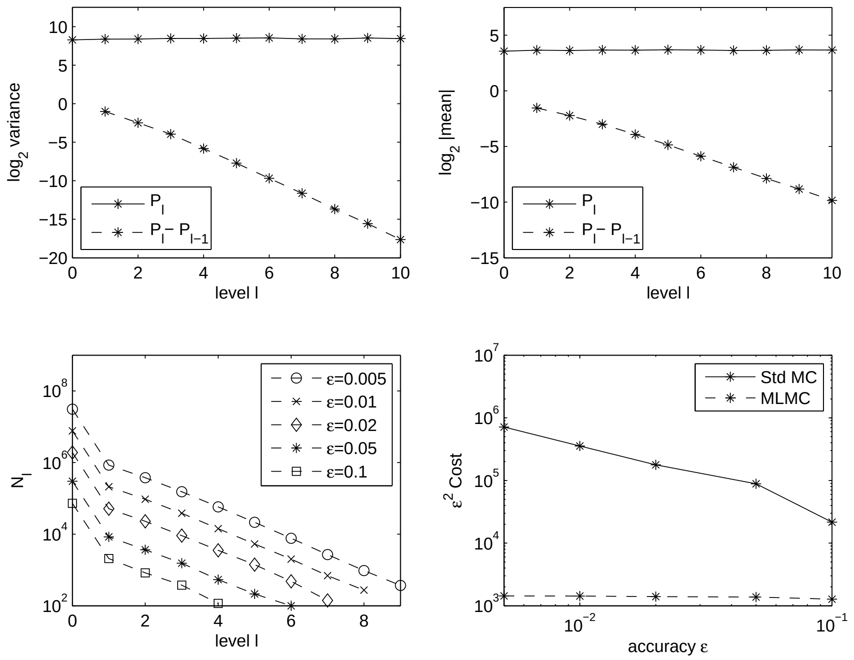

Fig. 1 European call option with constant Poisson rate

The bottom two plots correspond to five different multilevel calculations with different user-specified accuracies to be achieved. These use the numerical algorithm given in [Gil08] to determine the number of grid levels, and the optimal number of samples on each level, which are required to achieve the desired accuracy. The left plot shows that in each case many more samples are used on level 0 than on any other level, with very few samples used on the finest level of resolution. The right plot shows that the the multilevel cost is approximately proportional to ε^{-2}, which agrees with the computational complexity bound in Theorem 1 for the $\beta > 1$ case.

4.2 Lookback option

The payoff of the lookback option we consider is

$$P = \exp(-rT)\left(S(T) - \min_{0\le t\le T} S(t)\right).$$

Previous work [Gil07] achieved a second order convergence rate for the multilevel correction variance using the Milstein discretisation and an estimator constructed by approximating the behaviour within a timestep as an Itô process with constant drift and volatility, conditional on the endpoint values $\widehat{S}_n$ and $\widehat{S}_{n+1}$. Brownian Bridge results (see section 6.4 in [Gla04]) give the minimum value within the timestep $[t_n, t_{n+1}]$, conditional on the end values, as

$$\widehat{S}_{n,min} = \frac{1}{2}\left(\widehat{S}_n + \widehat{S}_{n+1} - \sqrt{\left(\widehat{S}_{n+1} - \widehat{S}_n\right)^2 - 2\, b_n^2\, h \log U_n}\,\right), \tag{6}$$

where b_n is the constant volatility and U_n is a uniform random variable on $[0,1]$. The same treatment can be used for the jump-adapted discretisation in this paper, except that $\widehat{S}^-_{n+1}$ must be used in place of $\widehat{S}_{n+1}$ in (6).

Equation (6) is used for the fine path approximation, but a different treatment is used for the coarse path, as in [Gil07]. This involves a change to the original telescoping sum in (4) which now becomes

$$\mathbb{E}[\widehat{P}^f_L] = \mathbb{E}[\widehat{P}^f_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}^f_\ell - \widehat{P}^c_{\ell-1}], \tag{7}$$

where $\widehat{P}^f_\ell$ is the approximation on level ℓ when it is the finer of the two levels being considered, and $\widehat{P}^c_\ell$ is the approximation when it is the coarser of the two. This modified telescoping sum remains valid provided $\mathbb{E}[\widehat{P}^f_\ell] = \mathbb{E}[\widehat{P}^c_\ell]$.

Considering a particular timestep in the coarse path construction, we have two possible situations. If it does not contain one of the fine path discretisation times, and therefore corresponds exactly to one of the fine path timesteps, then it is treated in the same way as the fine path, using the same uniform random number U_n. This leads naturally to a very small difference in the respective minima for the two paths.

The more complicated case is the one in which the coarse timestep contains one of the fine path discretisation times t', and so corresponds to the union of two fine path timesteps. In this case, the value at time t' is given by the conditional Brownian interpolant

$$\widehat{S}(t') = \widehat{S}_n + \mu\,(\widehat{S}^-_{n+1} - \widehat{S}_n) + b_n\left(W(t') - W_n - \mu\,(W_{n+1} - W_n)\right), \tag{8}$$

where $\mu = (t' - t_n)/(t_{n+1} - t_n)$ and the value of $W(t')$ comes from the fine path simulation. Given this value for $\widehat{S}(t')$, the minimum values for $S(t)$ within the two intervals $[t_n, t']$ and $[t', t_{n+1}]$ can be simulated in the same way as before, using the same uniform random numbers as the two fine timesteps.

The equality $\mathbb{E}[\widehat{P}^f_\ell] = \mathbb{E}[\widehat{P}^c_\ell]$ is respected in this treatment because $W(t')$ comes from the correct distribution, conditional on W_{n+1}, W_n, and therefore, conditional on the values of the Brownian path at the set of coarse discretisation points, the computed value for the coarse path minimum has exactly the same distribution as it would have if the fine path algorithm were applied.

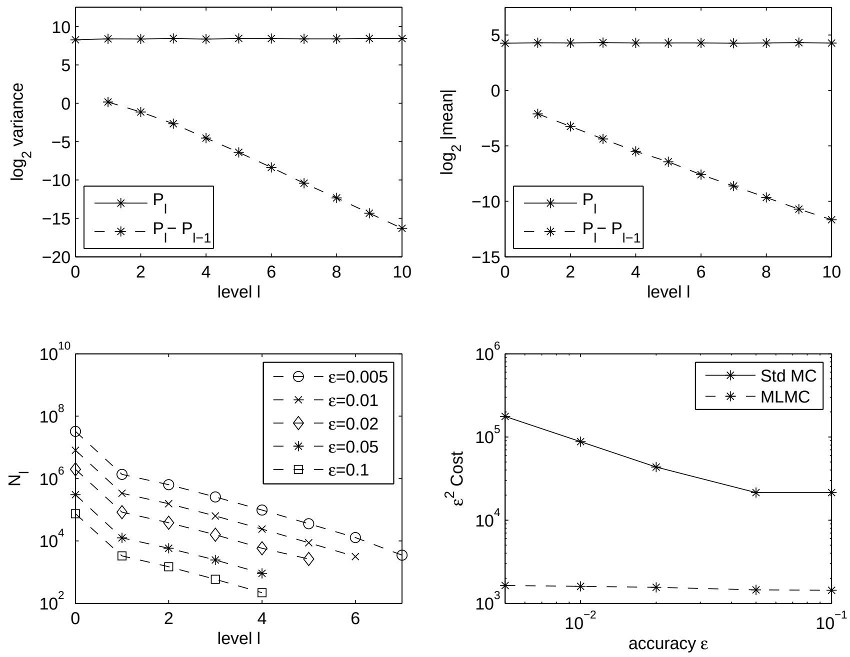

Fig. 2 Lookback option with constant Poisson rate

Further discussion and analysis of this is given in [XG11], including a proof that the strong error between the analytic path and the conditional interpolation approximation is at worst $O(h\log h)$.

Figure 2 presents the numerical results. The results are very similar to those obtained by Giles for geometric Brownian motion [Gil07]. The top two plots indicate second order variance convergence rate and first order weak convergence, both of which are consistent with the $O(h\log h)$ strong convergence. The computational cost of the multilevel method is therefore proportional to ε^{-2}, as shown in the bottom right plot.

4.3 Barrier option

We consider a down-and-out call barrier option for which the discounted payoff is

$$P = \exp(-rT)\,(S(T)-K)^{+}\,\mathbb{1}_{\{M_T>B\}},$$

where $M_T = \min_{0\le t\le T} S(t)$. The jump-adapted Milstein discretisation with the Brownian interpolation gives the approximation

$$\widehat{P} = \exp(-rT)\,(\widehat{S}(T)-K)^+\,\mathbb{1}_{\{\widehat{M}_T > B\}}$$

where $\widehat{M}_T = \min_{0\le t\le T}\widehat{S}(t)$. This could be simulated in exactly the same way as the lookback option, but in this case the payoff is a discontinuous function of the minimum M_T and an $O(h)$ error in approximating M_T would lead to an $O(h)$ variance for the multilevel correction.

Instead, following the approach of Cont & Tankov (see page 177 in [CT04]), it is better to use the expected value conditional on the values of the discrete Brownian increments and the jump times and magnitudes, all of which may be represented collectively as $\mathscr{F}$. This yields

$$\begin{aligned}
&\mathbb{E}\left[\exp(-rT)\;(\widehat{S}(T)-K)^+\,\mathbb{1}_{\{\widehat{M}_T>B\}}\right] \\
&\qquad = \mathbb{E}\left[\exp(-rT)\;(\widehat{S}(T)-K)^+\,\mathbb{E}\left[\mathbb{1}_{\{\widehat{M}_T>B\}}\mid\mathscr{F}\right]\right] \\
&\qquad = \mathbb{E}\left[\exp(-rT)\;(\widehat{S}(T)-K)^+\prod_{n=0}^{n_T-1}\widehat{p}_n\right]
\end{aligned}$$

where n_T is the number of timesteps, and $\widehat{p}_n$ denotes the conditional probability that the path does not cross the barrier B during the n^{th} timestep:

$$\widehat{p}_n = 1-\exp\left(\frac{-2\,(\widehat{S}_n-B)^+(\widehat{S}^-_{n+1}-B)^+}{b_n^2\;(t_{n+1}-t_n)}\right). \tag{9}$$

This barrier crossing probability is computed by conditional expectation and can be used to deduce the (6).

For the coarse path calculation, we again deal separately with two cases. When the coarse timestep does not include a fine path time, then we again use (9). In the other case, when it includes a fine path time t' we evaluate the Brownian interpolant at t' and then use the conditional expectation to obtain

$$\begin{aligned}
\widehat{p}_n = &\left\{1-\exp\left(\frac{-2\,(\widehat{S}_n-B)^+(\widehat{S}(t)-B)^+}{b_n^2\;(t'-t_n)}\right)\right\} \\
&\times\left\{1-\exp\left(\frac{-2\,(\widehat{S}(t')-B)^+(\widehat{S}^-_{n+1}-B)^+}{b_n^2\;(t_{n+1}-t')}\right)\right\}.
\end{aligned} \tag{10}$$

Figure 3 shows the numerical results for $K=100$, $B=85$. The top left plot shows that the multilevel variance is $O(h_\ell^\beta)$ for $\beta \approx 3/2$. This is similar to the behavior for a diffusion process [Gil07]. The bottom right plot shows that the computational cost of the multilevel method is again almost perfectly proportional to ε^{-2}.

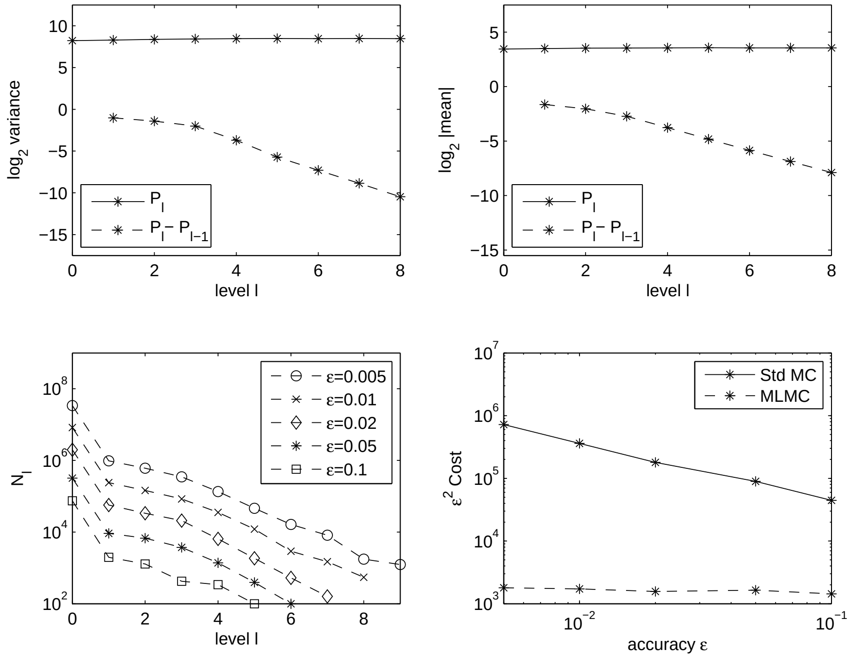

Fig. 3 Barrier option with constant Poisson rate

5 Path-dependent rates

In the case of a path-dependent jump rate $\lambda(S_t,t)$, the implementation of the multilevel method becomes more difficult because the coarse and fine path approximations may jump at different times. These differences could lead to a large difference between the coarse and fine path payoffs, and hence greatly increase the variance of the multilevel correction. To avoid this, we modify the simulation approach of Glasserman & Merener [GM04] which uses "thinning" to treat the case in which λ is bounded.

The idea of the thinning method is to construct a Poisson process with a constant rate $\lambda_{\sup}$ which is an upper bound of the state-dependent rate. This gives a set of candidate jump times, and these are then selected as true jump times with probability $\lambda(S_t,t)/\lambda_{\sup}$. Hence we have the following jump-adapted thinning Milstein scheme:

1. Generate the jump-adapted time grid for a Poisson process with constant rate $\lambda_{\sup}$;
2. Simulate each timestep using the Milstein discretisation;

3. When the endpoint t_{n+1} is a candidate jump time, generate a uniform random number $U \sim [0,1]$, and if $U < p_{t_{n+1}} = \dfrac{\lambda(S(t_{n+1}-),t_{n+1})}{\lambda_{\sup}}$, then accept t_{n+1} as a real jump time and simulate the jump.

5.1 Multilevel treatment

In the multilevel implementation, if we use the above algorithm with different acceptance probabilities for fine and coarse level, there may be some samples in which a jump candidate is accepted for the fine path, but not for the coarse path, or vice versa. Because of first order strong convergence, the difference in acceptance probabilities will be $O(h)$, and hence there is an $O(h)$ probability of coarse and fine paths differing in accepting candidate jumps. Such differences will give an $O(1)$ difference in the payoff value, and hence the multilevel variance will be $O(h)$. A more detailed analysis of this is given in [XG11].

To improve the variance convergence rate, we use a change of measure so that the acceptance probability is the same for both fine and coarse paths. This is achieved by taking the expectation with respect to a new measure Q:

$$\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}] = \mathbb{E}_Q[\widehat{P}_\ell \prod_\tau R^f_\tau - \widehat{P}_{\ell-1} \prod_\tau R^c_\tau]$$

where τ are the jump times. The acceptance probability for a candidate jump under the measure Q is defined to be $\frac{1}{2}$ for both coarse and fine paths, instead of $p_\tau = \lambda(S(\tau-),\tau)\,/\,\lambda_{\sup}$. The corresponding Radon-Nikodym derivatives are

$$R^f_\tau = \begin{cases} 2p^f_\tau, & \text{if } U < \dfrac{1}{2}\,; \\ 2(1-p^f_\tau), & \text{if } U \geq \dfrac{1}{2}\,, \end{cases} \qquad R^c_\tau = \begin{cases} 2p^c_\tau, & \text{if } U < \dfrac{1}{2}\,; \\ 2(1-p^c_\tau), & \text{if } U \geq \dfrac{1}{2}\,, \end{cases}$$

Since $R^f_\tau - R^c_\tau = O(h)$ and $\widehat{P}_\ell - \widehat{P}_{\ell-1} = O(h)$, this results in the multilevel correction variance $\mathbb{V}_Q[\widehat{P}_\ell \prod_\tau R^f_\tau - \widehat{P}_{\ell-1} \prod_\tau R^c_\tau]$ being $O(h^2)$.

If the analytic formulation is expressed using the same thinning and change of measure, the weak error can be decomposed into two terms as follows:

$$\mathbb{E}_Q\left[\widehat{P}_\ell \prod_\tau R^f_\tau - P \prod_\tau R_\tau\right] \;=\; \mathbb{E}_Q\left[(\widehat{P}_\ell - P)\, \prod_\tau R^f_\tau\right] \;+\; \mathbb{E}_Q\left[P\, (\prod_\tau R^f_\tau - \prod_\tau R_\tau)\right].$$

Using Hölder's inequality, the bound $\max(R_\tau, R^f_\tau) \leq 2$ and standard results for a Poisson process, the first term can be bounded using weak convergence results for the constant rate process, and the second term can be bounded using the corre-

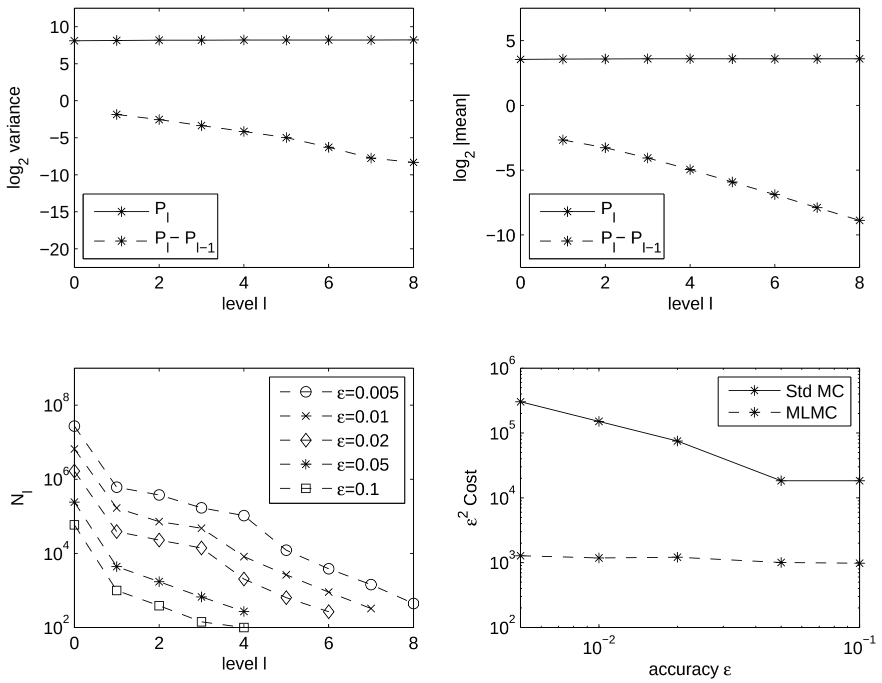

Fig. 4 European call option with path-dependent Poisson rate using thinning without a change of measure

sponding strong convergence results [XG11]. This guarantees that the multilevel procedure does converge to the correct value.

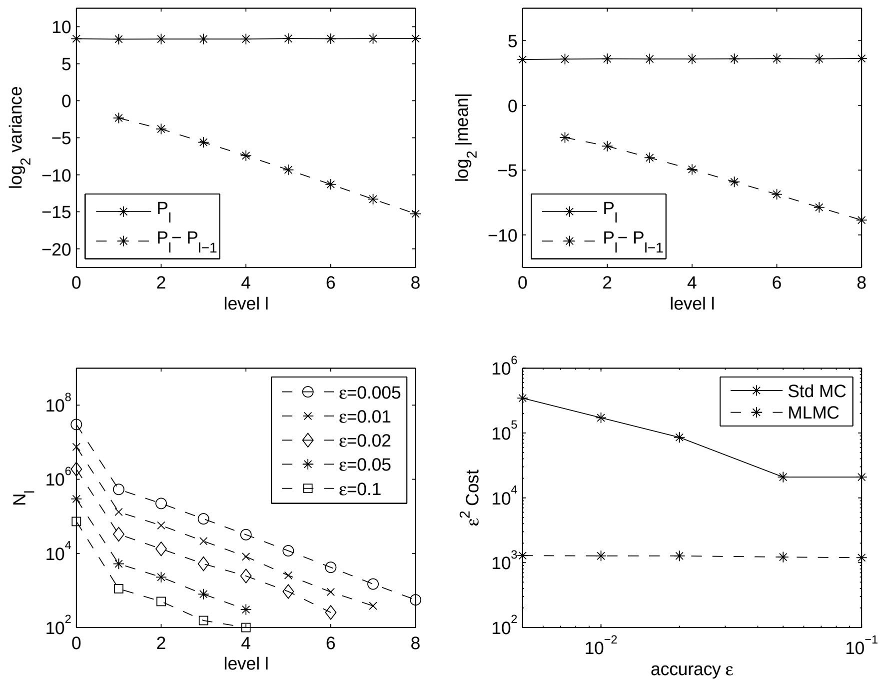

Fig. 5 European call option with path-dependent Poisson rate using thinning with a change of measure

5.1.1 Numerical results

We show numerical results for a European call option using

$$\lambda = \frac{1}{1+(S(t-)/S_0)^2}, \quad \lambda_{\sup} = 1,$$

and with all other parameters as used previously for the constant rate cases.

Comparing Figures 4 and 5 we see that the variance convergence rate is significantly improved by the change of measure, but there is little change in the computational cost. This is due to the main computational effort being on the coarsest level, which suggests using quasi-Monte Carlo on that level [GW09].

The bottom left plot in Figure 4 shows a slightly erratic behaviour. This is because the $O(h_\ell)$ variance is due to a small fraction of the paths having an $O(1)$ value for $\widehat{P}_\ell - \widehat{P}_{\ell-1}$. In the numerical procedure, the variance is estimated using an initial sample of 100 paths. When the variance is dominated by a few outliers, this sample size is not sufficient to provide an accurate estimate, leading to this variability.

6 Conclusions and future work

In this work we have extended the multilevel Monte Carlo method to scalar jump-diffusion SDEs using a jump-adapted discretisation. Second order variance convergence is maintained in the constant rate case for European options with Lipschitz payoffs, and also for lookback options by constructing estimators using a previous Brownian interpolation technique. Variance convergence of order 1.5 is obtained for barrier and digital options, which again matches the convergence which has been achieved previously for scalar SDEs without jumps. In the state-dependent rate case, we use thinning with a change of measure to avoid asynchronous jumps in the fine and coarse levels. In separate work [Xia11] we have also investigated an alternative approach using a time-change Poisson process to handle cases in which there is no upper bound on the jump rate.

The first natural direction for future work is numerical analysis to determine the order of convergence of multilevel correction variance [XG11]. A second is to investigate other Lévy processes, such as VG (Variance-Gamma), and NIG (Normal Inverse Gaussian). We also plan to investigate whether the multilevel quasi-Monte Carlo method [GW09] will further reduce the cost.

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