Numerical Methods II

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Euler-Maruyama method

For the vector SDE

$$\mathrm{d}S = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W$$

the Euler-Maruyama approximation is again

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\,h + b(\widehat{S}_n, t_n)\,\Delta W_n$$

but now $a(S,t)$ is a vector, $b(S,t)$ is a matrix and ΔW_n is a vector of Brownian increments with a prescribed correlation.

Remember: can define $\Delta W_n = L\,Z_n$ where elements of Z_n are i.i.d. unit Normals and

$$L\,L^T = \text{Cov}(\Delta W_n)$$

Euler-Maruyama method

Provided a and b are Lipschitz continuous in S:

- $O(h)$ weak convergence
- $O(\sqrt{h})$ strong convergence

Theoretical result (Clark & Cameron, 1980) proves this is the best strong convergence that can be achieved in the general vector case based solely on Brownian increments ΔW_n.

However, can do better in the scalar case, and also in the vector case in special cases or if we have additional information about the driving Brownian motion

Milstein Method

Start with scalar case:

$$\mathrm{d}S = a(S,t)\ \mathrm{d}t + b(S,t)\ \mathrm{d}W$$

which corresponds to the integral equation:

$$S(t) = S(0) + \int_0^t a(S(t),t)\,\mathrm{d}t + \int_0^t b(S(t),t)\,\mathrm{d}W(t)$$

where second integral is an Itô integral.

Approximating this on interval $[0,h]$ using

$$a(S(t),t) \approx a(S(0),0), \qquad b(S(t),t) \approx b(S(0),0)$$

gives Euler-Maruyama method.

Milstein Method

An asymptotic expansion gives

$$S(t) = S(0) + b(S(0),0)\, W(t) + O(h)$$

and hence

$$\begin{aligned} b(S(t),t)) &= b(S(0),0) + b'(S(0),0)\,(S(t)-S(0)) + O(h) \\ &= b(S(0),0) + b'(S(0),0)\; b(S(0),0)\; W(t) + O(h) \end{aligned}$$

This then leads to

$$S(h) = S(0) + a_0\, h + b_0\, W(h) + b'_0\, b_0 \int_0^h W(t)\, \mathrm{d}W(t) \; + O(h^{3/2})$$

where $a_0,\ b_0,\ b'_0$ are all evaluated at $(S(0),0)$.

Milstein Method

With a standard Lebesgue integral we would say

$$\int_0^h W(t)\,\mathrm{d}W(t) = \int_0^h \mathrm{d}(\tfrac{1}{2}W^2(t)) = \tfrac{1}{2}W^2(h)$$

but that would be wrong here!

Instead we must use Itô calculus to give

$$\mathrm{d}(\tfrac{1}{2}W^2) = W(t)\,\mathrm{d}W(t) + \tfrac{1}{2}\,\mathrm{d}t$$

and hence

$$\int_0^h W(t)\,\mathrm{d}W(t) = \tfrac{1}{2}\left(W^2(h) - h\right)$$

Milstein Method

This then gives us

$$S(h) = S(0) + a_0\, h + b_0\, W(h) + \tfrac{1}{2} b_0'\, b_0 \left(W^2(h) - h\right) \; + O(h^{3/2})$$

and so the Milstein scheme is

$$\begin{aligned} \widehat{S}_{n+1} &= \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n \\ &\quad + \tfrac{1}{2}\, b'(\widehat{S}_n, t_n)\; b(\widehat{S}_n, t_n) \left(\Delta W_n^2 - h\right) \end{aligned}$$

Note that $\mathbb{E}[\Delta W_n^2 - h] = 0, \;\; \mathbb{V}[\Delta W_n^2 - h] = 2\, h^2$

so the r.m.s. effect of extra “Milstein term” after T/h timesteps is $O(\sqrt{h})$, correcting the $O(\sqrt{h})$ error of the Euler-Maruyama method.

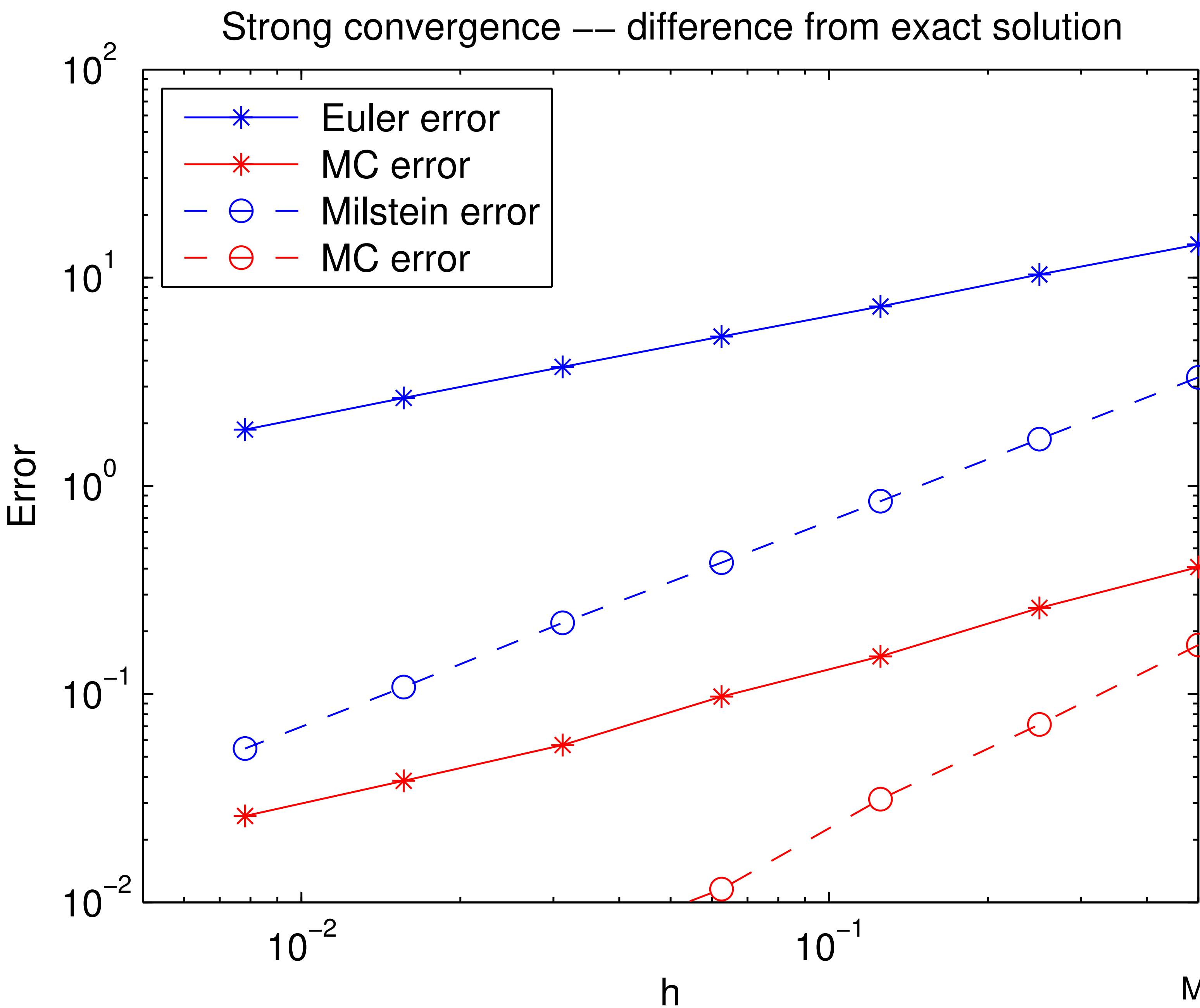
Strong convergence

Numerical demonstration: Geometric Brownian Motion

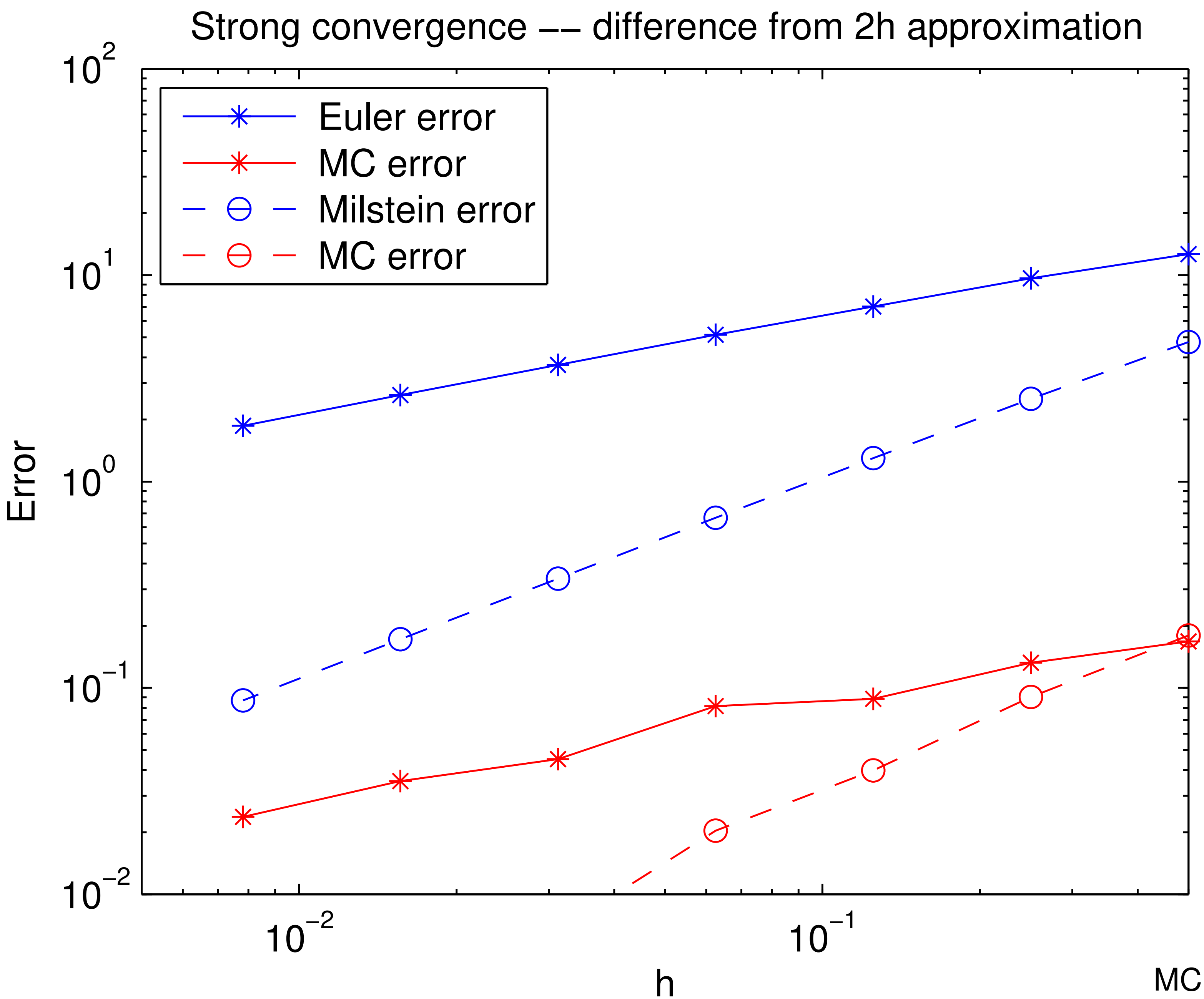
$$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W$$

$r = 0.05,\ \sigma = 0.5,\ T = 1$

Strong convergence



Strong convergence



Milstein Method

In the vector case, the SDE

$$\mathrm{d}S_i = a_i(S,t)\ \mathrm{d}t + \sum_j b_{ij}(S,t)\ \mathrm{d}W_j$$

corresponds to the integral equation:

$$S_i(t) = S_i(0) + \int_0^t a_i(S(t),t)\,\mathrm{d}t + \sum_j \int_0^t b_{ij}(S(t),t)\,\mathrm{d}W_j(t)$$

and the Euler-Maruyama approximation is

$$\widehat{S}_{i,n+1} = \widehat{S}_{i,n} + a_i(\widehat{S}_n,t_n)\,h + \sum_j b_{ij}(\widehat{S}_n,t_n)\,\Delta W_{j,n}$$

Milstein Method

An asymptotic expansion gives

$$S_i(t) \approx S_i(0) + \sum_j b_{ij}(S(0),0)\, W_j(t)$$

and hence

$$\begin{aligned} b_{ij}(S(t),t)) &\approx& b_{ij} + \sum_l \frac{\partial b_{ij}}{\partial S_l}\,(S_l(t) - S_l(0)) \\ &\approx& b_{ij} + \sum_{k,l} \frac{\partial b_{ij}}{\partial S_l}\, b_{lk}\, W_k(t) \end{aligned}$$

with b and its derivatives evaluated at $(S(0),0)$.

Milstein Method

This then leads to

$$S_i(h) \approx S_i(0) + a_i\, h + \sum_j b_{ij}\, W_j(h) + \sum_{j,k,l} \frac{\partial b_{ij}}{\partial S_l}\, b_{lk} \int_0^h W_k(t)\, \mathrm{d}W_j(t)$$

where a, b and its derivatives all evaluated at $(S(0), 0)$.

The problem now is to evaluate the iterated Itô integral

$$I_{jk} \equiv \int_0^h W_k(t)\, \mathrm{d}W_j(t)$$

Lévy areas

Itô calculus gives us

$$\mathrm{d}(W_j\, W_k) = W_j\ \mathrm{d}W_k(t) + W_k\ \mathrm{d}W_j(t) + \rho_{jk}\ \mathrm{d}t$$

where ρ_{jk} is the correlation between $\mathrm{d}W_j$ and $\mathrm{d}W_k$.

Hence,

$$W_j(h)\, W_k(h) - \rho_{jk}\, h = I_{kj} + I_{jk}$$

If we define the Lévy area to be

$$A_{jk} \ = \ I_{kj} - I_{jk} \ = \ \int_0^h W_j(t)\ \mathrm{d}W_k(t) - W_k(t)\ \mathrm{d}W_j(t)$$

then

$$I_{jk} = \tfrac{1}{2}\left(W_j(h)\, W_k(h) - \rho_{jk}\, h - A_{jk}\right)$$

Lévy areas

The problem is that there is no easy way to simulate the Lévy areas:

- conditional distribution depends on $W_j(h)$ and $W_k(t)$ so can't simply invert a cumulative distribution function
- Lyons & Gaines have an efficient technique in 2-dimensions but for higher dimensions, need to simulate Brownian motion within each timestep to approximate the Lévy area

However

- not all applications require it
- can be simulated very efficiently using many-core GPUs

Milstein method

The Milstein method is

$$
\begin{aligned}
\widehat{S}_{i,n+1} &= \widehat{S}_{i,n} + a_{i,n}\, h + \sum_j b_{ij,n}\, \Delta W_{j,n} \\
&\quad + \tfrac{1}{2} \sum_{j,k,l} \frac{\partial b_{ij}}{\partial S_l}\, b_{lk,n} \left(\Delta W_{j,n}\, \Delta W_{k,n} - \rho_{jk}\, h - A_{jk,n} \right)
\end{aligned}
$$

with

$$
A_{jk,n} = \int_{t_n}^{t_{n+1}} (W_j(t) - W_j(t_n))\, \mathrm{d}W_k(t) - (W_k(t) - W_k(t_n))\, \mathrm{d}W_j(t)
$$

Milstein method

However, using $A_{jk} = -A_{kj}$,

$$
\begin{aligned}
\sum_{j,k,l} \frac{\partial b_{ij}}{\partial S_l}\, b_{lk}\, A_{jk,n} &= -\sum_{j,k,l} \frac{\partial b_{ij}}{\partial S_l}\, b_{lk}\, A_{kj,n} \\
&= -\sum_{j,k,l} \frac{\partial b_{ik}}{\partial S_l}\, b_{lj}\, A_{jk,n} \\
&= \tfrac{1}{2} \sum_{j,k,l} \left(\frac{\partial b_{ij}}{\partial S_l}\, b_{lk} - \frac{\partial b_{ik}}{\partial S_l}\, b_{lj} \right) A_{jk,n}
\end{aligned}
$$

and so the Lévy areas are not need <u>if</u>, for all i, j, k,

$$
\sum_{l} \frac{\partial b_{ij}}{\partial S_l}\, b_{lk} - \frac{\partial b_{ik}}{\partial S_l}\, b_{lj} = 0.
$$

Milstein method

If b is a non-singular diagonal matrix, so each component of $S(t)$ is driven by a separate component of $W(t)$, the commutativity condition reduces to

$$\frac{\partial b_{ij}}{\partial S_k}\, b_{kk} - \frac{\partial b_{ik}}{\partial S_j}\, b_{jj} = 0$$

- if either $i\!=\!j\!=\!k$, or $i\!\neq\!j$ and $i\!\neq\!k$, this is satisfied
- if $i\!=\!j$ and $i\!\neq\!k$, it requires $\dfrac{\partial b_{ii}}{\partial S_k} = 0$
- if $i\!=\!k$ and $i\!\neq\!j$, it requires $\dfrac{\partial b_{ii}}{\partial S_j} = 0$
- hence, OK provided b_{ii} depends only on S_i

Final words

- Milstein scheme gives improved $O(h)$ strong convergence
- in vector case, needs simulation of Lévy areas in many cases, but not in some important applications
- weak convergence is <u>not</u> improved, so no benefit in most applications
- however, improved strong convergence does help with Multilevel Monte Carlo method
- other higher order methods have similar terms
  – the Euler-Maruyama method has the highest order of accuracy achievable using just Brownian increments