Numerical Methods II

Prof. Mike Giles

mike.giles@maths.ox.ac.uk

Oxford University Mathematical Institute

Greeks

In this lecture we will explore 2 more approaches:

- likelihood ratio method
- pathwise sensitivities

Likelihood ratio method

Defining $p(S)$ to the probability density function for the final state $S(T)$, then

$$V = \mathbb{E}[f(S(T))] = \int f(S)\, p(S)\, \mathrm{d}S,$$

$$\Longrightarrow \quad \frac{\partial V}{\partial \theta} = \int f\, \frac{\partial p}{\partial \theta}\, \mathrm{d}S = \int f\, \frac{\partial (\log p)}{\partial \theta}\, p\, \mathrm{d}S = \mathbb{E}\left[f\, \frac{\partial (\log p)}{\partial \theta} \right]$$

The quantity $\dfrac{\partial (\log p)}{\partial \theta}$ is sometimes called the "score function".

Likelihood ratio method

Note that when $f = 1$, we get

$$\frac{\partial}{\partial \theta} \mathbb{E}[1] = 0$$

and therefore

$$\mathbb{E}\left[\frac{\partial (\log p)}{\partial \theta}\right] = 0$$

This is a handy check to make sure we have derived the score function correctly.

Likelihood ratio method

Example: GBM with arbitrary payoff $f(S(T))$.

For the usual Geometric Brownian motion with constants r, σ, the final log-normal probability distribution is

$$p(S) = \frac{1}{S\sigma\sqrt{2\pi T}} \; \exp\left[-\frac{1}{2} \left(\frac{\log(S/S_0) - (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} \right)^2 \right]$$

$$\log p = -\log S - \log\sigma - \tfrac{1}{2}\log(2\pi T) - \tfrac{1}{2} \frac{\left(\log(S/S_0) - (r - \frac{1}{2}\sigma^2)T\right)^2}{\sigma^2 T}$$

$$\Longrightarrow \quad \frac{\partial \log p}{\partial S_0} = \frac{\log(S/S_0) - (r - \frac{1}{2}\sigma^2)T}{S_0 \sigma^2 T}$$

Likelihood ratio method

Hence

$$\Delta = \mathbb{E}\left[\frac{\log(S/S_0) - (r - \frac{1}{2}\sigma^2)T}{S_0\,\sigma^2 T}\ f(S(T))\right]$$

In the Monte Carlo simulation,

$$\log(S/S_0) - (r - \tfrac{1}{2}\sigma^2)T = \sigma\, W(T)$$

so the expression can be simplified to

$$\Delta = \mathbb{E}\left[\frac{W(T)}{S_0\,\sigma\, T}\ f(S(T))\right]$$

– very easy to implement so you estimate Δ at the same time as estimating the price V

Likelihood ratio method

Similarly for vega we have

$$
\begin{aligned}
\frac{\partial \log p}{\partial \sigma} \;=\;& -\frac{1}{\sigma} - \frac{\log(S/S_0) - (r-\frac{1}{2}\sigma^2)T}{\sigma} \\
&+ \frac{\left(\log(S/S_0) - (r-\frac{1}{2}\sigma^2)T\right)^2}{\sigma^3 T}
\end{aligned}
$$

and hence

$$
\text{vega} \;= \mathbb{E}\left[\left(\frac{1}{\sigma}\left(\frac{W(T)^2}{T} - 1\right) - W(T)\right) f(S(T))\right]
$$

Likelihood ratio method

In both cases, the variance is very large when σ is small, and it is also large for Δ when T is small.

More generally, LRM is usually the approach with the largest variance.

Likelihood ratio method

To get second derivatives, note that

$$\frac{\partial^2 \log p}{\partial \theta^2} \;=\; \frac{\partial}{\partial \theta}\left(\frac{1}{p}\,\frac{\partial p}{\partial \theta}\right) \;=\; \frac{1}{p}\,\frac{\partial^2 p}{\partial \theta^2} - \frac{1}{p^2}\left(\frac{\partial p}{\partial \theta}\right)^2$$

$$\Longrightarrow \quad \frac{1}{p}\,\frac{\partial^2 p}{\partial \theta^2} \;=\; \frac{\partial^2 \log p}{\partial \theta^2} + \left(\frac{\partial \log p}{\partial \theta}\right)^2$$

and hence

$$\frac{\partial^2 V}{\partial \theta^2} = \mathbb{E}\left[\left(\frac{\partial^2 \log p}{\partial \theta^2} + \left(\frac{\partial \log p}{\partial \theta}\right)^2\right) f(S(T))\right]$$

Likelihood ratio method

In the multivariate extension, $X = \log S(T)$ can be written as

$$X = \mu + L\,Z$$

where μ is the mean vector, $\Sigma = L\,L^T$ is the covariance matrix and Z is a vector of uncorrelated Normals. The joint p.d.f. is

$$\log p = -\tfrac{1}{2}\log|\Sigma| - \tfrac{1}{2}(X-\mu)^T\Sigma^{-1}(X-\mu) - \tfrac{1}{2}d\,\log(2\pi).$$

and after a lot of algebra we obtain

$$\frac{\partial \log p}{\partial \mu} = L^{-T} Z,$$

$$\frac{\partial \log p}{\partial \Sigma} = \tfrac{1}{2}\,L^{-T}\left(Z Z^T - I\right)L^{-1}$$

Pathwise sensitivities

Start instead with

$$V \equiv \mathbb{E}\left[f(S(T))\right] = \int f(S(T))\; p_W(W)\; \mathrm{d}W$$

and differentiate this to get

$$\frac{\partial V}{\partial \theta} = \int \frac{\partial f}{\partial S}\; \frac{\partial S(T)}{\partial \theta}\; p_W \;\mathrm{d}W = \mathbb{E}\left[\frac{\partial f}{\partial S}\; \frac{\partial S(T)}{\partial \theta}\right]$$

with $\partial S(T)/\partial \theta$ being evaluated at fixed W.

Note: this derivation needs $f(S)$ to be differentiable, but by considering the limit of a sequence of smoothed (regularised) functions can prove it's OK provided $f(S)$ is continuous and piecewise differentiable

Pathwise sensitivities

This leads to the estimator

$$\frac{1}{N}\sum_{i=1}^{N} \frac{\partial f}{\partial S}(S^{(i)})\ \frac{\partial S^{(i)}}{\partial \theta}$$

which is the derivative of the usual price estimator

$$\frac{1}{N}\sum_{i=1}^{N} f(S^{(i)})$$

Gives incorrect estimates when $f(S)$ is discontinuous.
e.g. for digital put $\dfrac{\partial f}{\partial S} = 0$ so estimated value of Greek is zero – clearly wrong.

Pathwise sensitivities

Extension to second derivatives is straightforward

$$
\begin{aligned}
\frac{\partial^2 V}{\partial \theta^2} &= \int \left\{ \frac{\partial^2 f}{\partial S^2} \left(\frac{\partial S(T)}{\partial \theta} \right)^2 + \frac{\partial f}{\partial S}\, \frac{\partial^2 S(T)}{\partial \theta^2} \right\} p_W \, \mathrm{d}W \\
&= \mathbb{E} \left[\frac{\partial^2 f}{\partial S^2} \left(\frac{\partial S(T)}{\partial \theta} \right)^2 + \frac{\partial f}{\partial S}\, \frac{\partial^2 S(T)}{\partial \theta^2} \right]
\end{aligned}
$$

with $\partial^2 S(T)/\partial\theta^2$ also being evaluated at fixed W.

However, this requires $f(S)$ to have a continuous first derivative – a problem in practice

Pathwise sensitivities

Extension to multivariate case is straightforward

$$S_k(T) = S_k(0)\ \exp\left((r - \tfrac{1}{2}\sigma_k^2)T + \sum_l L_{kl} X_l \right)$$

so

$$\log S_k(T) = \log S_k(0) + (r - \tfrac{1}{2}\sigma_k^2)T + \sum_l L_{kl} X_l$$

and hence

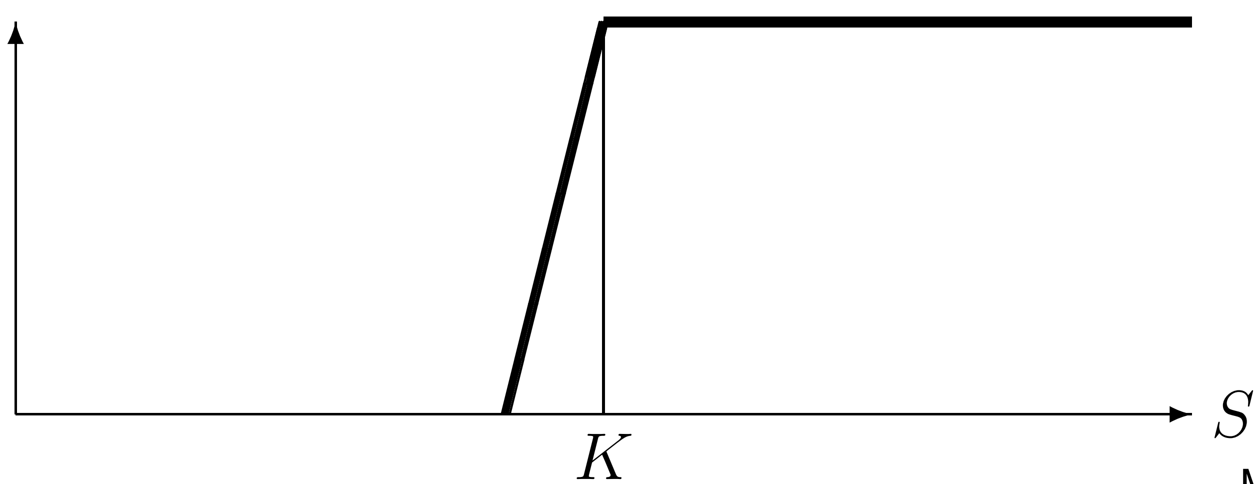
$$\frac{1}{S_k(T)}\ \frac{\partial S_k(T)}{\partial \theta} = \frac{1}{S_k(0)}\ \frac{\partial S_k(0)}{\partial \theta} - \sigma_k \frac{\partial \sigma_k}{\partial \theta} T + \sum_l \frac{\partial L_{kl}}{\partial \theta} X_l$$

Pathwise sensitivities

To handle payoffs which do not have the necessary continuity/smoothness one can modify the payoff

For digital options it is common to use a piecewise linear approximation to limit the magnitude of Δ near maturity – avoids large transaction costs

Bank selling the option will price it conservatively (i.e. over-estimate the price)



Pathwise sensitivities

The standard call option definition can be smoothed by integrating the smoothed Heaviside function

$$H_\varepsilon(S\!-\!K) = \Phi\left(\frac{S\!-\!K}{\varepsilon}\right)$$

with $\varepsilon \ll K$, to get

$$f(S) = (S\!-\!K)\ \Phi\left(\frac{S\!-\!K}{\varepsilon}\right) + \frac{\varepsilon}{\sqrt{2\pi}} \exp\left(-\ \frac{(S\!-\!K)^2}{2\,\varepsilon^2}\right)$$

This will allow the calculation of Γ and other second derivatives

Final Words

Estimating Greeks is an important task:

- LRM can handle discontinuous payoffs, but a little complicated for multivariate case, and will see next term that it doesn't extend well to path-dependent options
- pathwise sensitivity is usually the best approach (simplest, lowest variance and least cost) when it is applicable – needs continuous payoff for first derivatives
- payoff smoothing can be used to make pathwise approach applicable to discontinuous payoffs and for second derivatives
- alternatively, combine pathwise sensivity with finite differences for second derivatives – e.g. use pathwise to compute Δ, then finite difference this to get Γ