Numerical Methods II
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Problem sheet 3: solutions

1. A Taylor series expansion gives

$$\log(1{+}\varepsilon) = \varepsilon - \tfrac{1}{2}\varepsilon^2 + O(\varepsilon^3)$$

and hence

$$\log(1 + r\,h + \sigma\,\Delta W) = r\,h + \sigma\,\Delta W - \tfrac{1}{2}\sigma^2\,(\Delta W)^2 + O(h^{3/2})$$

and therefore

$$\widehat{X}_{T/h} \approx r\,T + \sigma\,W(T) - \sum_n \tfrac{1}{2}\sigma^2\,(\Delta W)^2.$$

The exact solution is

$$X(T) = (r{-}\tfrac{1}{2}\sigma^2)T + \sigma W(T)$$

and hence the error is

$$X(T) - \widehat{X}_{T/h} \approx \sum_n \tfrac{1}{2}\,\sigma^2\left((\Delta W_n)^2 - h\right).$$

Since the $(\Delta W_n)^2$ are independent with mean h and variance $2h^2$, the Central Limit Theorem tells us that in the limit $h\to 0$ the sum is Normally distributed with zero mean and variance $\frac{1}{2}\,\sigma^4 T\,h$.

2. If $a = r\,S,\ b = \sigma S$, the suggested numerical approximation gives

$$\begin{aligned}
\widehat{S}^{(p)}_{n+1} &= \widehat{S}_n(1 + r\,h + \sigma\,\Delta W_n) \\
\widehat{S}_{n+1} &= \widehat{S}_n\left(1 + r\,h + \sigma\,\Delta W_n + \tfrac{1}{2}r^2h^2 + r\,h\,\sigma\,\Delta W_n + \tfrac{1}{2}\sigma^2(\Delta W_n)^2\right)
\end{aligned}$$

and following the same approach as in the previous question we find that the error is

$$X(T) - \widehat{X}_{T/h} \approx \sum_n -\tfrac{1}{2}\sigma^2\,h = -\tfrac{1}{2}\sigma^2\,T$$

3. (a) A bit of a trick question as the Milstein method in this case is the same as the Euler-Maruyama method since the volatility does not depend on S. Hence, in this case the Euler method has first order strong convergence, which explains the numerical results obtained in the last practical.

   (b) In this case $b' = \frac{1}{2}\sigma S^{-1/2}$ and so the Milstein method is

$$\widehat{S}_{n+1} = \widehat{S}_n + \kappa\,(\theta{-}\widehat{S}_n)\,h + \sigma\sqrt{\widehat{S}_n}\;\Delta W_n + \tfrac{1}{4}\sigma^2\left((\Delta W_n)^2 - h\right)$$

On a practical note: there is a possibility that the numerical method could lead to $\widehat{S}_{n+1}$ being negative, in which case the square root becomes a problem for the next step. This is usually dealt with by modifying the square root to use

$$\sqrt{\widehat{S}_n^+} \equiv \sqrt{\max(\widehat{S}_n, 0)}.$$

4. The Euler-Maruyama method for Geometric Brownian Motion gives

$$\widehat{S}_{n+1} = \widehat{S}_n (1 + r\,h + \sigma\,\Delta W_n)$$

If we define

$$s_n^{(1)} = \frac{\partial \widehat{S}_n}{\partial S_0}, \qquad s_n^{(2)} = \frac{\partial \widehat{S}_n}{\partial \sigma}$$

then straightforward differentiation yields

$$\begin{aligned} s_{n+1}^{(1)} &= s_n^{(1)} (1 + r\,h + \sigma\,\Delta W_n), \\ s_{n+1}^{(2)} &= s_n^{(2)} (1 + r\,h + \sigma\,\Delta W_n) + \widehat{S}_n\,\Delta W_n \end{aligned}$$

with initial data $s_n^{(1)} = 1$, $s_n^{(2)} = 0$.

Now we have to consider each of the payoffs:

(a) For the barrier option we have

$$\widehat{f}(\widehat{S}) = \exp(-rT)\ (\widehat{S}_{T/h} - K)^+ \ \prod_n (1 - P_n).$$

with

$$P_n = \exp\left(- \frac{2\,(\widehat{S}_{n+1} - B)^+\,(\widehat{S}_n - B)^+}{\sigma^2\,\widehat{S}_n^2\ h} \right)$$

For Delta, differentiating these gives (when $\widehat{S}_{n+1} > B$ and $\widehat{S}_n > B$)

$$\begin{aligned} \frac{\partial \widehat{f}(\widehat{S})}{\partial S_0} &= \exp(-rT)\ s_{T/h}^{(1)} \mathbf{1}_{\widehat{S}_{T/h} - K} \ \prod_n (1 - P_n) \\ &\quad - \exp(-rT)\ (\widehat{S}_{T/h} - K)^+ \sum_n \left\{ \left(\prod_{m \neq n} (1 - P_m) \right) \frac{\partial P_n}{\partial S_0} \right\} \end{aligned}$$

with

$$\frac{\partial P_n}{\partial S_0} = - \left(\frac{2\,s_{n+1}^{(1)}\,(\widehat{S}_n - B) + 2\,(\widehat{S}_{n+1} - B)\,s_n^{(1)}}{\sigma^2\,\widehat{S}_n^2\ h} - \frac{4\,s_n^{(1)}\,(\widehat{S}_{n+1} - B)\,(\widehat{S}_n - B)}{\sigma^2\,\widehat{S}_n^3\ h} \right) P_n$$

The expression for Vega is similar, with $s_n^{(2)}$ instead of $s_n^{(1)}$, except that

$$\begin{aligned} \frac{\partial P_n}{\partial \sigma} &= - \left(\frac{2\,s_{n+1}^{(2)}\,(\widehat{S}_n - B) + 2\,(\widehat{S}_{n+1} - B)\,s_n^{(2)}}{\sigma^2\,\widehat{S}_n^2\ h} - \frac{4\,s_n^{(2)}\,(\widehat{S}_{n+1} - B)\,(\widehat{S}_n - B)}{\sigma^2\,\widehat{S}_n^3\ h} \right. \\ &\qquad \left. - \frac{4\,(\widehat{S}_{n+1} - B)\,(\widehat{S}_n - B)}{\sigma^3\,\widehat{S}_n^2\ h} \right) P_n \end{aligned}$$

(b) For the lookback option

$$\widehat{f}(\widehat{S}) = \exp(-rT) \ \left(\widehat{S}_{T/h} - \min_n \widehat{M}_n \right)$$

where

$$\widehat{M}_n = \tfrac{1}{2} \left(\widehat{S}_{n+1} + \widehat{S}_n - \sqrt{(\widehat{S}_{n+1} - \widehat{S}_n)^2 - 2\,\sigma^2\, \widehat{S}_n^2\, h\, \log U_n} \right)$$

If we let m be the timestep which gives the minimum (i.e. $\widehat{M}_m = \min_n \widehat{M}_n$) then Delta is given by

$$\frac{\partial \widehat{f}}{\partial S_0} = \exp(-rT) \ \left(s^{(1)}_{T/h} - \frac{\partial \widehat{M}_m}{\partial S_0} \right)$$

where

$$\frac{\partial \widehat{M}_m}{\partial S_0} = \tfrac{1}{2} \left(s^{(1)}_{m+1} + s^{(1)}_m - \frac{(\widehat{S}_{m+1} - \widehat{S}_m)(s^{(1)}_{m+1} - s^{(1)}_m) - 2\,\sigma^2\, \widehat{S}_m s^{(1)}_m\, h\, \log U_m}{\sqrt{(\widehat{S}_{m+1} - \widehat{S}_m)^2 - 2\,\sigma^2\, \widehat{S}_m^2\, h\, \log U_m}} \right)$$

The expression for Vega is similar, with $s^{(2)}_n$ instead of $s^{(1)}_n$, except that

$$\frac{\partial \widehat{M}_m}{\partial \sigma} = \tfrac{1}{2} \left(s^{(2)}_{m+1} + s^{(2)}_m - \frac{(\widehat{S}_{m+1} - \widehat{S}_m)(s^{(2)}_{m+1} - s^{(2)}_m) - 2\,(\sigma^2\, \widehat{S}_m s^{(2)}_m + \sigma\, \widehat{S}_m^2)\, h\, \log U_m}{\sqrt{(\widehat{S}_{m+1} - \widehat{S}_m)^2 - 2\,\sigma^2\, \widehat{S}_m^2\, h\, \log U_m}} \right)$$