# Assignment

## Part 1

1. Complete the analysis outlined in lecture 6, extending the strong error analysis to the more general case of $\mathbb{E}[\sup_{t\in[0,T]} |\widehat{S}_t - S_t|^p]$ for $p \geq 2$ for a scalar SDE.

2. If $X$ is a real, scalar, random variable with zero mean, the standard Markov inequality is $\mathbb{P}[|X| > c] \leq \mathbb{E}[|X|]/c$, for any $c > 0$, which generalises to $\mathbb{P}[|X| > c] \leq \mathbb{E}[|X|^p]/c^p$, for any $p \geq 1$.

   Suppose that $\overline{X}_N$ is an average of $N$ i.i.d. samples $X_n, n = 1, 2, \ldots, N$. For $p = 2$, we have $\mathbb{E}[\,\overline{X}_N^{\,2}] = N^{-1}\mathbb{E}[\,X^2]$, and hence $\mathbb{P}[\,|\overline{X}_N| > c\,] \leq \mathbb{E}[\,X^2] \,/\, (c^2 N)$. For larger values of $p$, for which $\mathbb{E}[\,|X|^p]$ is finite, we have the following lemma and corollary:

   **Lemma 0.1.** *For $p \geq 2$, if $\mathbb{E}[|X|^p]$ is finite then there exists a constant $C_p$, depending only on $p$, such that*

   $$\mathbb{E}[\,|\overline{X}_N|^p] \;\leq\; C_p \, N^{-p/2} \, \mathbb{E}\,[\,|X|^p]$$

   **Corollary 0.2.** *For $p \geq 2$, if $\mathbb{E}[\,|X|^p]$ is finite then there exists a constant $C_p$, depending only on $p$, such that*

   $$\mathbb{P}[\,|\overline{X}_N| > c\,] \;\leq\; C_p \, \mathbb{E}[\,|X|^p] \,/\, (c^2 N)^{p/2}.$$

   *Proof.* Follows immediately from the Lemma and the standard Markov inequality. □

   Note: this is different from the usual Central Limit Theorem which holds $c^2 N$ fixed as $N \to \infty$. Here we have a result which holds as $c^2 N \to \infty$.

   Your task: prove the Lemma using the BDG and Jensen inequalities.

# Part 2

Carry out a simple numerical experiment which you are free to choose to be relevant to your interests and/or research.

Possible examples include:

- investigate order of strong convergence of Euler-Maruyama and Milstein discretisations for one or more SDEs of your choice, or I can make some suggestions (lecture 2)
- investigate order of strong convergence of Euler-Maruyama discretisation using adaptive time-stepping, and look for errors in our numerical analysis for this case
- multilevel Monte Carlo estimation of an output functional based on the solution of a 2D elliptic PDE subject in a unit square subject to random boundary conditions and/or random coefficients in the PDE (lecture 11)
- multilevel Monte Carlo estimation of the point value of a simple 2D parabolic PDE through use of the Feynman-Kac formula and the simulation of stopping times for the relevant SDE (lecture 12)