Stochastic Numerical Analysis
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Exercise 1

Let U be uniformly distributed on $[0,1]$. You are to use Monte Carlo simulation to estimate the value of

$$\overline{f} = \mathbb{E}[f(U)] = \int_0^1 f(U)\ \mathrm{d}U$$

where

$$f(x) = x\ \cos \pi x.$$

1. Calculate analytically the exact value for $\overline{f}$ and

$$\sigma^2 = \mathbb{E}[(f(U)-\overline{f})^2] = \int_0^1 (f(U)-\overline{f})^2\ \mathrm{d}U$$

2. Using the Matlab `rand` function, write a Matlab program which computes

$$Y_m = N^{-1} \sum_{n=1}^{N} f(U^{(m,n)})$$

   for 1000 different sets of 1000 independent random variables $U^{(m,n)}$.

3. Sort the Y_m into ascending order, and then plot $C_m = (m-1/2)/1000$ versus Y_m – this is the numerical cumulative distribution function.

   Superimpose on the same plot the cumulative distribution function you would expect from the Central Limit Theorem (you may want to use the Matlab functions `normcdf` or `norminv`) and comment on your results.

   You may like to experiment by trying larger or smaller sets of points to improve your understanding of the asymptotic behaviour described by the CLT.

4. Modify your code to use a single set of 10^6 random numbers, and plot

$$Y_N = N^{-1} \sum_{n=1}^{N} f(U^{(n)})$$

   versus N for $N = 10^3 - 10^6$. This should demonstrate the convergence to the true value predicted by the Strong Law of Large Numbers.

   For each N, also compute an unbiased estimate for the variance σ^2 and hence add to the plot upper and lower confidence bounds based on 3 standard deviations of the variation in the mean.

   Add a line corresponding to the true value. Does this lie inside the bounds?