Mean-square stability analysis

Today we are looking at the numerical analysis in


What’s new?

- new numerical methods
- new definition and analysis of numerical stability
Stochastic theta method

The stochastic theta method for approximating the SDE

$$dX_t = f(X_t) \, dt + g(X_t) \, dW_t$$

is

$$\hat{X}_{n+1} = \hat{X}_n + (1-\theta) f(\hat{X}_n) \, h + \theta f(\hat{X}_{n+1}) \, h + g(\hat{X}_n) \, \Delta W_n$$

- $\theta = 0$ is Euler-Maruyama method
- $\theta = 1$ is drift-implicit method
- $\theta = 1/2$ is stochastic equivalent of Crank-Nicholson method
ODE stability analysis

The linear ODE:

\[ dX_t = -\lambda X_t \, dt \]

has a solution which decays exponentially if \( \lambda > 0 \).

The numerical approximation

\[ \hat{X}_{n+1} = \hat{X}_n - (1-\theta)\lambda h \hat{X}_n - \theta \lambda h \hat{X}_{n+1} \]

also decays exponentially if

\[ \left| \frac{1 - (1-\theta)\lambda h}{1 + \theta \lambda h} \right| < 1 \]

so need either \( \theta \geq 1/2 \), or \( \lambda h < 2/(1-2\theta) \).
What about the corresponding SDE?

\[ dX_t = -\lambda X_t \, dt + \mu X_t \, dW_t \]

First of all, how does the analytic solution behave?

Itô calculus gives us

\[ dX_t^2 = -2(\lambda - \frac{1}{2} \mu^2) X_t^2 \, dt + 2\mu X_t^2 \, dW_t \]

and hence

\[ d \left( \mathbb{E}[X_t^2] \right) = -2 \left( \lambda - \frac{1}{2} \mu^2 \right) \mathbb{E}[X_t^2] \, dt \]

so \( \mathbb{E}[X_t^2] \) decays exponentially if \( \lambda > \frac{1}{2} \mu^2 \)
SDE stability analysis

The numerical approximation is

\[ \hat{X}_{n+1} = \hat{X}_n - (1-\theta) \lambda h \hat{X}_n - \theta \lambda h \hat{X}_{n+1} + \mu \hat{X}_n \Delta W_n \]

Setting \( \Delta W_n = h^{1/2} Z_n \) where \( Z_n \) is a standard Normal r.v., we can re-arrange to get

\[ \hat{X}_{n+1} = (a + b Z_n) \hat{X}_n \]

where

\[ a = \frac{1 - (1-\theta) \lambda h}{1 + \theta \lambda h}, \quad b = \frac{\mu h^{1/2}}{1 + \theta \lambda h} \]
SDE stability analysis

Since

\[ \hat{X}^2_{n+1} = (a + b Z_n)^2 \hat{X}^2_n \]

it follows that

\[ \mathbb{E}[\hat{X}^2_{n+1}] = (a^2 + b^2) \mathbb{E}[\hat{X}^2_n] \]

so \( \mathbb{E}[\hat{X}^2_n] \) decays exponentially iff \( a^2 + b^2 < 1 \), which corresponds to

\[ (1 - 2 \theta) \lambda^2 h < 2 (\lambda - \frac{1}{2} \mu^2) \]

If \( \lambda - \frac{1}{2} \mu^2 > 0 \), then it’s unconditionally stable for \( \theta \geq 1/2 \), while for \( \theta < 1/2 \) the timestep stability limit is

\[ h < \frac{2 (\lambda - \frac{1}{2} \mu^2)}{1 - 2 \theta} \]
SDE stability analysis

Generalisation to vector systems – the algorithm

\[ \hat{X}_{n+1} = (A + Z_n B) \hat{X}_n \]

with scalar \( Z_n \), vector \( \hat{X}_n \) and matrices \( A, B \), leads to

\[ \hat{X}_{n+1}^T \hat{X}_{n+1} = \hat{X}_n^T (A + Z_n B)^T (A + Z_n B) \hat{X}_n \]

and hence

\[ \mathbb{E} \left[ \hat{X}_{n+1}^T \hat{X}_{n+1} \right] = \mathbb{E} \left[ \hat{X}_n^T (A^T A + B^T B) \hat{X}_n \right] \]

\( A^T A + B^T B \) is symmetric and positive (semi-)definite, so a sufficient condition for mean-square stability is that the largest eigenvalue is less than 1.
SDE stability analysis

It can be further generalised to

\[ \hat{X}_{n+1} = (A + Z_n B + W_n C) \hat{X}_n \]

where \( W_n \) is an additional, independent Normal r.v.

This leads to

\[ \mathbb{E} \left[ \hat{X}_{n+1}^T \hat{X}_{n+1} \right] = \mathbb{E} \left[ \hat{X}_n^T (A^T A + B^T B + C^T C) \hat{X}_n \right] \]

with a similar test for mean-square stability.
Parabolic SPDE

Unusual parabolic SPDE arises in CDO modelling (Bush, Hambly, Haworth & Reisinger)

\[ dp = -\mu \frac{\partial p}{\partial x} \, dt + \frac{1}{2} \frac{\partial^2 p}{\partial x^2} \, dt + \sqrt{\rho} \frac{\partial p}{\partial x} \, dW \]

with absorbing boundary \( p(0, t) = 0 \)

- derived in limit as number of firms \( \rightarrow \infty \)
- \( x \) is distance to default
- \( p(x, t) \) is probability density function
- \( dW \) term corresponds to systemic risk
- \( \frac{\partial^2 p}{\partial x^2} \) comes from idiosyncratic risk
Parabolic SPDE

Milstein and central difference discretisation leads to

\[ p_{j}^{n+1} = p_{j}^{n} - \frac{\mu k + \sqrt{\rho} k Z_n}{2h} (p_{j+1}^{n} - p_{j-1}^{n}) + \frac{(1-\rho) k + \rho k Z_n^2}{2h^2} (p_{j+1}^{n} - 2p_{j}^{n} + p_{j-1}^{n}) \]

where \( k \) is the timestep, \( h \) is the uniform grid spacing, and \( Z_n \sim N(0, 1) \),

Considering a Fourier mode

\[ p_{j}^{n} = g_n \exp(i j \theta), \quad |\theta| \leq \pi \]

leads to …
Parabolic SPDE

\[ g_{n+1} = \left( a(\theta) + b(\theta) Z_n + c(\theta) Z_n^2 \right) g_n, \]

where

\[ a(\theta) = 1 - \frac{i \mu k}{h} \sin \theta - \frac{2(1 - \rho) k}{h^2} \sin^2 \frac{\theta}{2}, \]

\[ b(\theta) = -\frac{i \sqrt{\rho} k}{h} \sin \theta, \]

\[ c(\theta) = -\frac{2 \rho k}{h^2} \sin^2 \frac{\theta}{2}. \]
Parabolic SPDE

Following Higham’s mean-square stability analysis approach,

\[ \mathbb{E}[|g_{n+1}|^2] = \mathbb{E} \left[ (a + b Z_n + c Z_n^2)(a^* + b^* Z_n + c^* Z_n^2) \ |g_n|^2 \right] \]

\[ = (|a+c|^2 + |b|^2 + 2|c|^2) \ \mathbb{E}[|g_n|^2] \]

so stability requires \(|a+c|^2 + |b|^2 + 2|c|^2 \leq 1\) for all \(\theta\),

which leads to a timestep stability limit:

\[ \mu^2 k \leq 1 - \rho, \]

\[ \frac{k}{h^2} \leq (1 + 2\rho^2)^{-1}. \]
Parabolic SPDE

This can be extended to finite domains using matrix stability analysis, writing the discrete equations as

\[ P_{n+1} = (A + B Z_n + C Z_n^2) P_n, \quad \text{where} \]

\[ A = I - \frac{\mu k}{2h} D_1 + \frac{(1-\rho) k}{2h^2} D_2, \quad B = -\frac{\sqrt{\rho k}}{2h} D_1, \quad C = \frac{\rho k}{2h^2} D_2, \]

and \( D_1 \) and \( D_2 \) look like

\[ D_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -2 & 1 \\ 1 & -2 & 1 \\ 1 & -2 & 1 \end{pmatrix}. \]
Parabolic SPDE

\[ \mathbb{E}[P_{n+1}^T P_{n+1}] = \mathbb{E}[P_n^T (A^T + B^T Z_n + C^T Z_n^2)(A + B Z_n + C Z_n^2) P_n] \]

\[ = \mathbb{E}[P_n^T ((A+C)^T (A+C) + B^T B + 2 C^T C) P_n] \]

\(D_1\) is anti-symmetric and \(D_2\) is symmetric, and

\[ D_1 D_2 - D_2 D_1 = E_1 - E_2, \quad D_1^2 = D_3 + E_1 + E_2 \]

where \(D_3\) looks like

\[ D_3 = \begin{pmatrix} -3 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix} \]
and \( E_1 \) and \( E_2 \) are zero apart from one corner element,

\[
E_1 = \begin{pmatrix} 2 \\ \end{pmatrix}, \quad E_2 = \begin{pmatrix} \ \ \\ 2 \end{pmatrix}
\]

This leads to

\[
\mathbb{E} \left[ V_n^T \left( (A+C)^T (A+C) + B^T B + 2 C^T C \right) V_n \right] \\
= \mathbb{E} \left[ V_n^T M V_n \right] - (e_1 + e_2) \mathbb{E}[(v_1^n)^2] - (e_1 - e_2) \mathbb{E}[(v_{j-1}^n)^2],
\]

where \( e_1 \) and \( e_2 \) are scalars and

\[
M = I - \frac{k}{h^2} D_2 + \frac{k^2}{4 h^4} D_2^2 - \left( \frac{\rho k}{4 h^2} + \frac{\mu^2 k^2}{4 h^2} \right) D_3.
\]
It can be verified that the $m^{th}$ eigenvector of $M$ is a Fourier mode and the associated eigenvalue is

$$|a(\theta_m) + c(\theta_m)|^2 + |b(\theta_m)|^2 + 2|c(\theta_m)|^2$$

where $a(\theta)$, $b(\theta)$, $c(\theta)$ are the same functions as before.

In the limit $h, k/h \to 0$, $e_1 \pm e_2 > 0$, and therefore the Fourier stability condition

$$\sup_{\theta} \left\{ |a(\theta) + c(\theta)|^2 + |b(\theta)|^2 + 2|c(\theta)|^2 \right\} \leq 1$$

is also a sufficient condition for mean-square matrix stability.
This turns out to be a good application for multilevel MC:

- coarsest level of approximation uses 1 timestep per quarter, and 10 spatial points
- each finer level uses four times as many timesteps, and twice as many spatial points – ratio is due to numerical stability constraints
- computational cost $C_\ell \propto 8^\ell$
- numerical results suggest variance $V_\ell \propto 8^{-\ell}$
- can prove $V_\ell \propto 16^{-\ell}$ when no absorbing boundary
Parabolic SPDE

Fractional loss on equity tranche of a 5-year CDO:

\[
\log_2 \text{variance} = P_l - P_{l-1}
\]

\[
\log_2 |\text{mean}| = P_l - P_{l-1}
\]
Parabolic SPDE

Fractional loss on equity tranche of a 5-year CDO:

![Graph showing the relationship between level l and accuracy ε, with different lines for different ε values. The graph on the right shows the ε^2 Cost as a function of accuracy ε.](image)

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