Stochastic Numerical Analysis

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Strong error analysis

We are now ready to carry out the strong error analysis for the Euler-Maruyama method applied to scalar SDEs.

This follows the analysis in Kloeden & Platen, and assumes the SDE satisifies “the usual conditions”:

- Lipschitz continuity in space:

  $$|a(x,t) - a(y,t)| + |b(x,t) - b(y,t)| \leq K\ |x-y|$$

- linear growth bound:

  $$|a(x,t)| + |b(x,t)| \ \leq\ K\ (1+|x|)$$

- square-root continuity in time:

  $$|a(x,s) - a(x,t)| + |b(x,s) - b(x,t)| \ \leq\ K\ (1+|x|)\sqrt{|s-t|}$$

SDE bounds

First, we need an additional result about the SDE solution.

Theorem: (Thm 4.5.4 on page 136 of Kloeden & Platen)
Under the standard conditions, when starting at time t_0 from a random starting value S_{t_0}, then for any $p \geq 2$ there are constants C, D such that

$$\mathbb{E}(|S_t - S_{t_0}|^p) \leq D \;\left(1 + \mathbb{E}(|S_{t_0}|^p)\right)\; (t - t_0)^{p/2}\; e^{C(t - t_0)}$$

Proof: a slightly more careful version of the proof that the moments are bounded.

In particular, when $t - t_0 < h$, this implies that

$$\mathbb{E}(|S_t - S_{t_0}|^2) \leq c\, h$$

Strong error analysis

The Euler-Maruyama discretisation is

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n$$

This defines the numerical approximation at a discrete set of times, but what about within each timestep?

Not really necessary to define a numerical approximation at intermediate times, but convenient for the analysis, and can be useful in some applications (lecture 10?)

Strong error analysis

Define $\underline{t}$ to be t rounded down to nearest discrete time t_n

$$\underline{t} = \max_n \{t_n : t_n \leq t\}$$

Then the continuous-time E-M approximation is given by

$$\widehat{S}_t = \widehat{S}_0 + \int_0^t a(\widehat{S}_{\underline{s}}, \underline{s})\, \mathrm{d}s + \int_0^t b(\widehat{S}_{\underline{s}}, \underline{s})\, \mathrm{d}W_s$$

Bounded moments

Another preliminary theorem.

Theorem: Under the standard conditions, starting from a fixed S_0, for any $T > 0$ and any $p \geq 2$,

$$\mathbb{E}\left[\sup_{[0,T]} |\widehat{S}_t|^p\right] \; < \; C_{p,T}$$

where $C_{p,T}$ does not depend on h. i.e. all moments of $\sup_{s\in[0,t]} |\widehat{S}_s|$ are uniformly bounded.

Proof: essentially the same as the proof of the bounded moments for the SDE solution S_t.

Bounded moments

We now come to the main theorem, which is Thm 10.2.2 in Kleoden & Platen.

Theorem: Under the standard conditions, starting from a fixed S_0, for any $T>0$, there is a constant C which depends only on T, K such that

$$\mathbb{E}\left[\sup_{[0,T]} |S_t - \widehat{S}_t|^2\right] \ \leq \ C\ h$$

Strong error analysis

We want to determine a bound of the form

$$Z(T) \;\equiv\; \mathbb{E}\left[\sup_{[0,T]} E_t^2\right] \;\leq\; C\, h$$

where the error $E_t \equiv S_t - \widehat{S}_t$ satisfies

$$E_t = \int_0^t a(S_s, s) - a(\widehat{S}_{\underline{s}}, \underline{s})\ \mathrm{d}s + \int_0^t b(S_s, s) - b(\widehat{S}_{\underline{s}}, \underline{s})\ \mathrm{d}W_s$$

As usual, our aim is to construct an integral inequality

$$Z(t) \leq \alpha + \beta \int_0^t Z(s)\ \mathrm{d}s$$

and then use Grönwall's inequality.

Strong error analysis

Note that

$$a(S_s, s) - a(\widehat{S}_{\underline{s}}, \underline{s}) = A_1 + A_2 + A_3$$

where

$$\begin{aligned} A_1 &= a(S_s, s) - a(S_s, \underline{s}), && \text{which depends on } s - \underline{s} \\ A_2 &= a(S_s, \underline{s}) - a(S_{\underline{s}}, \underline{s}), && \text{which depends on } S_s - S_{\underline{s}} \\ A_3 &= a(S_{\underline{s}}, \underline{s}) - a(\widehat{S}_{\underline{s}}, \underline{s}), && \text{which depends on } S_{\underline{s}} - \widehat{S}_{\underline{s}} \end{aligned}$$

Strong error analysis

There is a similar decomposition for $b(S_s,s)-b(\widehat{S}_{\underline{s}},\underline{s})$, so

$$E_t = \int_0^t (A_1+A_2+A_3)\,\mathrm{d}s + \int_0^t (B_1+B_2+B_3)\,\mathrm{d}W_s$$

and Jensen's inequality then gives us

$$\begin{aligned}\mathbb{E}\left[\sup_{[0,t]} E_s^2\right] &\leq 6\sum_{j=1}^{3}\mathbb{E}\left[\sup_{[0,t]}\left(\int_0^s A_j\,\mathrm{d}u\right)^2\right]\\ &+ 6\sum_{j=1}^{3}\mathbb{E}\left[\sup_{[0,t]}\left(\int_0^s B_j\,\mathrm{d}W_u\right)^2\right]\end{aligned}$$

Strong error analysis

Note that for $0 < s < t < T$

$$\left(\int_0^s A_j \, \mathrm{d}u \right)^2 \; \leq \; T \int_0^s A_j^2 \, \mathrm{d}u \; \leq \; T \int_0^t A_j^2 \, \mathrm{d}u$$

so

$$\mathbb{E}\left[\sup_{[0,t]} \left(\int_0^s A_j \, \mathrm{d}u \right)^2 \right] \; \leq \; T \int_0^t \mathbb{E}[A_j^2] \, \mathrm{d}u$$

Also, Doob's inequality, plus the Itô isometry, gives us

$$\mathbb{E}\left[\sup_{[0,t]} \left(\int_0^s B_j \, \mathrm{d}W_u \right)^2 \right] \; \leq \; 4 \int_0^t \mathbb{E}[B_j^2] \, \mathrm{d}u$$

Strong error analysis

We now look separately at each of the 6 integrals.

For the first one, we use the square-root continuity in time to give

$$\int_0^t \mathbb{E}[A_1^2]\,\mathrm{d}s \;\leq\; \int_0^t h\,K^2\,\mathbb{E}[(1+|S_s|)^2]\,\mathrm{d}s \;\leq\; C_1\,h$$

since $\displaystyle\int_0^T \sup_{[0,t]} |S_s|^2 \mathrm{d}s < \infty$

A similar bound applies to $\displaystyle\int_0^t \mathbb{E}[B_1^2]\,\mathrm{d}s$

Strong error analysis

Looking at the second integral, Lipschitz continuity in space gives

$$\int_0^t \mathbb{E}[A_2^2]\,\mathrm{d}s \;\leq\; \int_0^t K^2\,\mathbb{E}[(S_s - S_{\underline{s}})^2]\,\mathrm{d}s \;\leq\; C_2\,h$$

since

$$\int_0^T \mathbb{E}\left[\,|S_s - S_{\underline{s}}|^2\right]\,\mathrm{d}s \;\leq\; c\,h$$

A similar bound applies to $\displaystyle\int_0^t \mathbb{E}[B_2^2]\,\mathrm{d}s$

Strong error analysis

Finally, again due to Lipschitz continuity, the third integral has bound

$$\int_0^t \mathbb{E}[A_3^2]\,\mathrm{d}s \;\leq\; \int_0^t K^2\,\mathbb{E}[(\widehat{S}_{\underline{s}}-S_{\underline{s}})^2]\,\mathrm{d}s \;\leq\; K^2\int_0^t Z(s)\,\mathrm{d}s$$

with a similar bound for $\displaystyle\int_0^t \mathbb{E}[B_3^2]\,\mathrm{d}s$

This gives us all of the ingredients to obtain the desired integral inequality, with $\alpha = O(h)$, and hence, by Grönwall's inequality, the final result.

Extensions

This has proved that

$$\mathbb{E}\left[|S_t - \widehat{S}_t|^2\right] = O(h)$$

It can be generalised to prove that

$$\mathbb{E}\left[|S_t - \widehat{S}_t|^p\right] = O(h^{p/2})$$

Key change: switch to BDG inequality for stochastic integrals.

Extensions

Assignment for everyone (MSc and DPhil): work through the details and write it up.

You can assume (as I have done)

- standard results for SDEs (in particular the additional one I have given you today)
- the fact that $\mathbb{E}\left[\sup_{[0,T]} |\widehat{S}_t|^p\right] < \infty$ for all $p \geq 2$

but should justify carefully the rest of the analysis.

Extensions

The analysis can also be generalised to vector SDEs

This doesn't introduce any fundamental new difficulties, just lots of subscripts for all of the different vector and matrix components.

Extensions

Another extension is to higher order methods such as the Milstein scheme. This requires additional smoothness of $a(S,t)$ and $b(S,t)$, with bounds on higher derivatives.

The continuous numerical approximation is defined as

$$\begin{aligned}\widehat{S}_t &= \widehat{S}_0 + \int_0^t a(\widehat{S}_{\underline{s}}, \underline{s})\,\mathrm{d}s + \int_0^t b(\widehat{S}_{\underline{s}}, \underline{s})\,\mathrm{d}W_s \\ &\quad + \int_0^t b'(\widehat{S}_{\underline{s}}, \underline{s})\, b(\widehat{S}_{\underline{s}}, \underline{s})\,(W_s - W_{\underline{s}})\,\mathrm{d}W_s\end{aligned}$$

Main tricky bit (I think) is to analyse the difference

$$b(S_t, \underline{t}) - \Big(b(S_{\underline{t}}, \underline{t}) + b'(S_{\underline{t}}, \underline{t})\; b(S_{\underline{t}}, \underline{t})\; (W_t - W_{\underline{t}}) \Big)$$