Multilevel quasi-Monte Carlo path simulation

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Acknowledgments: Frances Kuo, Ian Sloan, Ben Waterhouse (UNSW – 2007)

and Adrien Grumberg (Oxford – 2015)

Outline

Objective of this research was faster Monte Carlo simulation of path dependent options to estimate values and Greeks.

Several separate ingredients:

- multilevel method
- quasi-Monte Carlo
- adjoint pathwise Greeks
- parallel computing on NVIDIA graphics cards

Emphasis in this presentation was on multilevel QMC

Generic Problem

Stochastic differential equation with general drift and volatility terms:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t)$$

We want to compute the expected value of an option dependent on $S(t)$. In the simplest case of European options, it is a function of the terminal state

$$P = f(S(T))$$

with a uniform Lipschitz bound,

$$|f(U) - f(V)| \;\leq c\,\|U-V\|\,, \quad \forall\, U,V.$$

Simplest MC Approach

Euler discretisation with timestep h:

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n$$

Estimator for expected payoff is an average of N independent path simulations:

$$\widehat{Y} = N^{-1} \sum_{i=1}^{N} f(\widehat{S}^{(i)}_{T/h})$$

- weak convergence – $O(h)$ error in expected payoff
- strong convergence – $O(h^{1/2})$ error in individual path

Simplest MC Approach

Mean Square Error is $O\left(N^{-1} + h^2\right)$

- first term comes from variance of estimator
- second term comes from bias due to weak convergence

To make this $O(\varepsilon^2)$ requires

$$N = O(\varepsilon^{-2}), \quad h = O(\varepsilon) \quad \Longrightarrow \quad \text{cost} = O(N\, h^{-1}) = O(\varepsilon^{-3})$$

Aim is to improve this cost to $O\left(\varepsilon^{-p}\right)$, with p as small as possible, ideally close to 1.

Note: for a relative error of $\varepsilon = 0.001$, the difference between ε^{-3} and ε^{-1} is huge.

Standard MC Improvements

- variance reduction techniques (e.g. control variates, stratified sampling) improve the constant factor in front of ε^{-3}, sometimes spectacularly
- improved second order weak convergence (e.g. through Richardson extrapolation) leads to $h = O(\sqrt{\varepsilon})$, giving $p = 2.5$
- quasi-Monte Carlo reduces the number of samples required, at best leading to $N \approx O(\varepsilon^{-1})$, giving $p \approx 2$ with first order weak methods

Multilevel method gives $p = 2$ without QMC, and at best $p \approx 1$ with QMC.

MLMC Approach

Consider multiple sets of simulations with different timesteps $h_\ell = 2^{-\ell}\, T,\ \ell = 0, 1, \ldots, L$, and payoff $\widehat{P}_\ell$

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$$

Expected value is same – aim is to reduce variance of estimator for a fixed computational cost.

Key point: approximate $\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ using N_ℓ simulations with $\widehat{P}_\ell$ and $\widehat{P}_{\ell-1}$ obtained using <u>same</u> Brownian path.

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{i=1}^{N_\ell} \left(\widehat{P}_\ell^{(i)} - \widehat{P}_{\ell-1}^{(i)} \right)$$

MLMC Approach

Using independent paths for each level, the variance of the combined estimator is

$$\mathbb{V}\left[\sum_{\ell=0}^{L} \widehat{Y}_\ell\right] = \sum_{\ell=0}^{L} N_\ell^{-1} V_\ell, \qquad V_\ell \equiv \mathbb{V}[\widehat{P}_\ell - \widehat{P}_{\ell-1}],$$

and the computational cost is proportional to $\displaystyle\sum_{\ell=0}^{L} N_\ell\, h_\ell^{-1}$.

Hence, the variance is minimised for a fixed computational cost by choosing N_ℓ to be proportional to $\sqrt{V_\ell\, h_\ell}$.

The constant of proportionality can be chosen so that the combined variance is $O(\varepsilon^2)$.

Multilevel MC Approach

Theorem: *Let P be a functional of the solution of a stochastic o.d.e., and $\widehat{P}_\ell$ the discrete approximation using a timestep $h_\ell = M^{-\ell}\, T$.*

If *there exist independent estimators $\widehat{Y}_\ell$ based on N_ℓ Monte Carlo samples, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that*

i) $\mathbb{E}[\widehat{P}_\ell - P] \leq c_1\, h_\ell^\alpha$

ii) $\mathbb{E}[\widehat{Y}_\ell] = \begin{cases} \mathbb{E}[\widehat{P}_0], & l = 0 \\ \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & l > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_l] \leq c_2\, N_l^{-1} h_l^\beta$

iv) C_ℓ*, the computational complexity of* $\widehat{Y}_\ell$*, is bounded by*

$$C_\ell \leq c_3\, N_\ell\, h_\ell^{-1}$$

Multilevel MC Approach

then *there exists a positive constant* c_4 *such that for any* $\varepsilon < e^{-1}$ *there are values* L *and* N_l *for which the multi-level estimator*

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell,$$

has Mean Square Error $MSE \equiv \mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$

with a computational complexity C *with bound*

$$C \leq \begin{cases} c_4\,\varepsilon^{-2}, & \beta > 1, \\ c_4\,\varepsilon^{-2}(\log \varepsilon)^2, & \beta = 1, \\ c_4\,\varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

Milstein Scheme

The theorem suggests use of Milstein scheme —
better strong convergence, same weak convergence

Generic scalar SDE:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t), \quad 0<t<T.$$

Milstein scheme:

$$\widehat{S}_{n+1} = \widehat{S}_n + a\,h + b\,\Delta W_n + \tfrac{1}{2}\,b'\,b\left((\Delta W_n)^2 - h\right).$$

Milstein Scheme

In scalar case:

- $O(h)$ strong convergence
- $O(\varepsilon^{-2})$ complexity for Lipschitz payoffs – trivial
- $O(\varepsilon^{-2})$ complexity for Asian, lookback, barrier and digital options using carefully constructed estimators based on Brownian interpolation or extrapolation

Milstein Scheme

Brownian interpolation: within each timestep, model the behaviour as simple Brownian motion conditional on the two end-points

$$\begin{aligned}\widehat{S}(t) &= \widehat{S}_n + \lambda(t)(\widehat{S}_{n+1} - \widehat{S}_n) \\ &\quad + b_n \Big(W(t) - W_n - \lambda(t)(W_{n+1} - W_n) \Big),\end{aligned}$$

where

$$\lambda(t) = \frac{t - t_n}{t_{n+1} - t_n}$$

There then exist analytic results for the distribution of the min/max/average over each timestep.

Milstein Scheme

Brownian extrapolation for final timestep:

$$\widehat{S}_N = \widehat{S}_{N-1} + a_{N-1}\, h + b_{N-1}\, \Delta W_N$$

– considering all possible ΔW_N gives Gaussian distribution, for which a digital option has a known conditional expectation (Glasserman)

This payoff smoothing can be generalised to multivariate cases, and leads to a “vibrato” Monte Carlo technique which is suitable for both efficient multilevel analysis and the computation of Greeks

Results

Geometric Brownian motion:

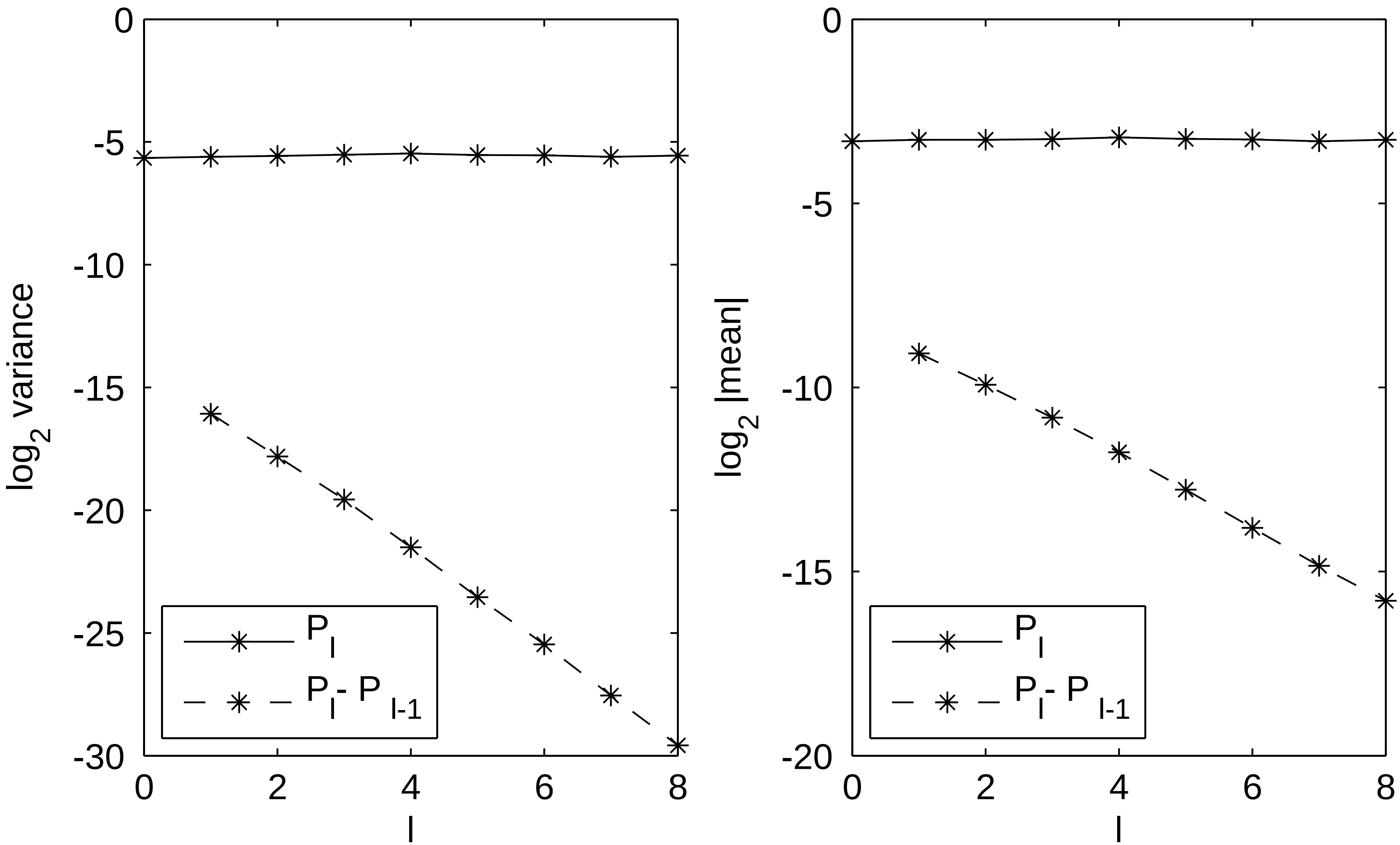
$$\mathrm{d}S = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W, \qquad 0 < t < T,$$

with parameters $T\!=\!1,\ \ S(0)\!=\!1,\ \ r\!=\!0.05,\ \ \sigma\!=\!0.2$

- European call option: $\exp(-rT)\ \max(S(T)-1,0)$
- European digital call: $\exp(-rT)\,\mathbf{1}_{S(T)>1}$
- Down-and-out barrier option: same as call provided $S(t)$ stays above $B\!=\!0.9$

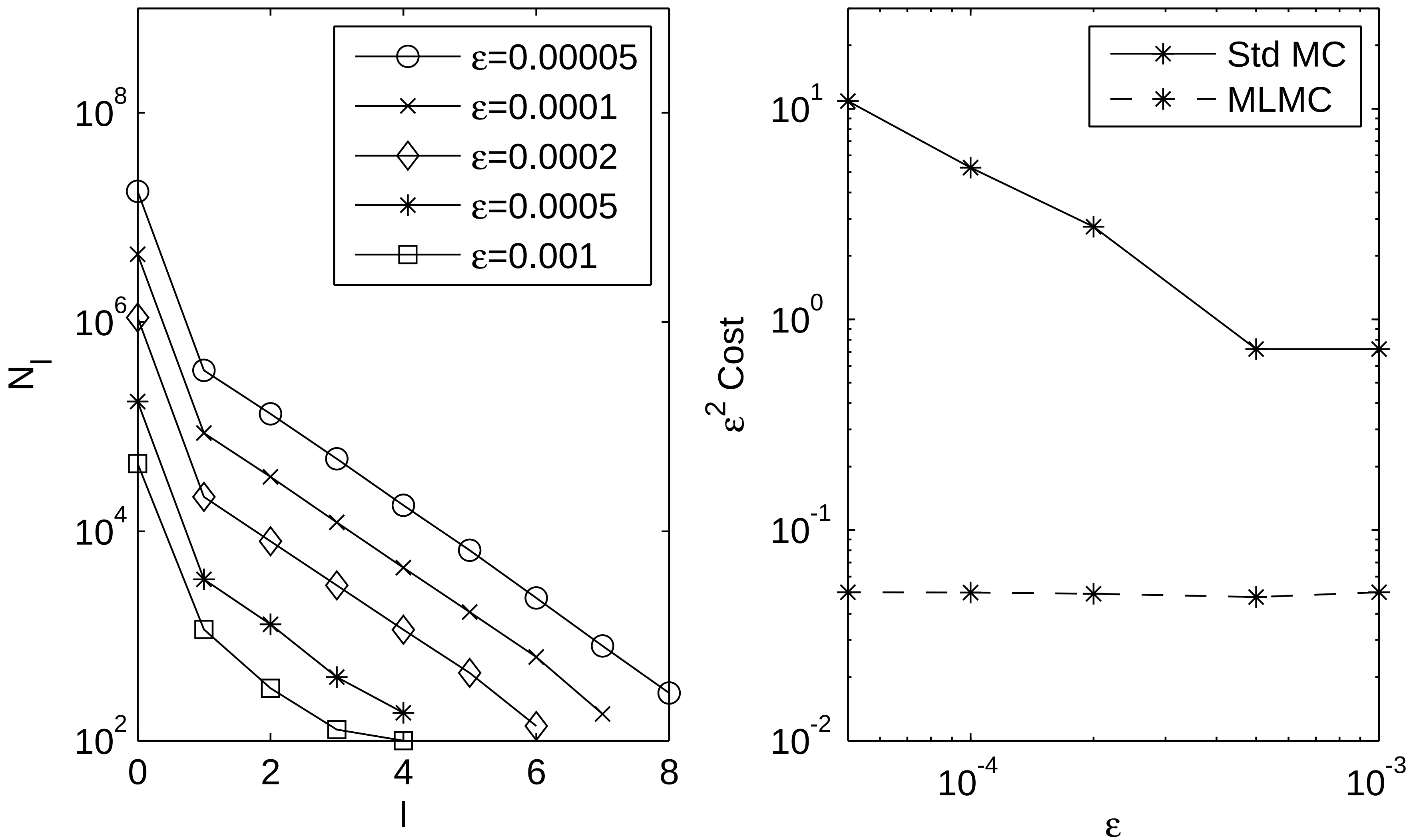
MLMC Results

GBM: European call



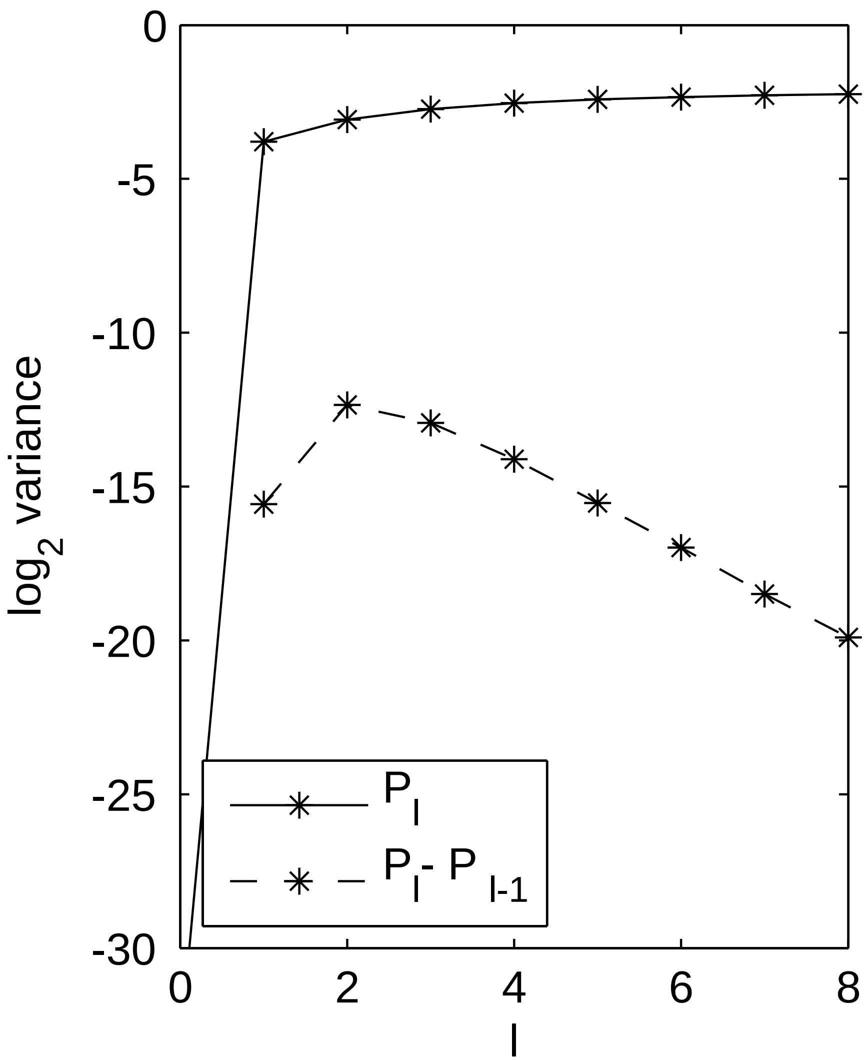
MLMC Results

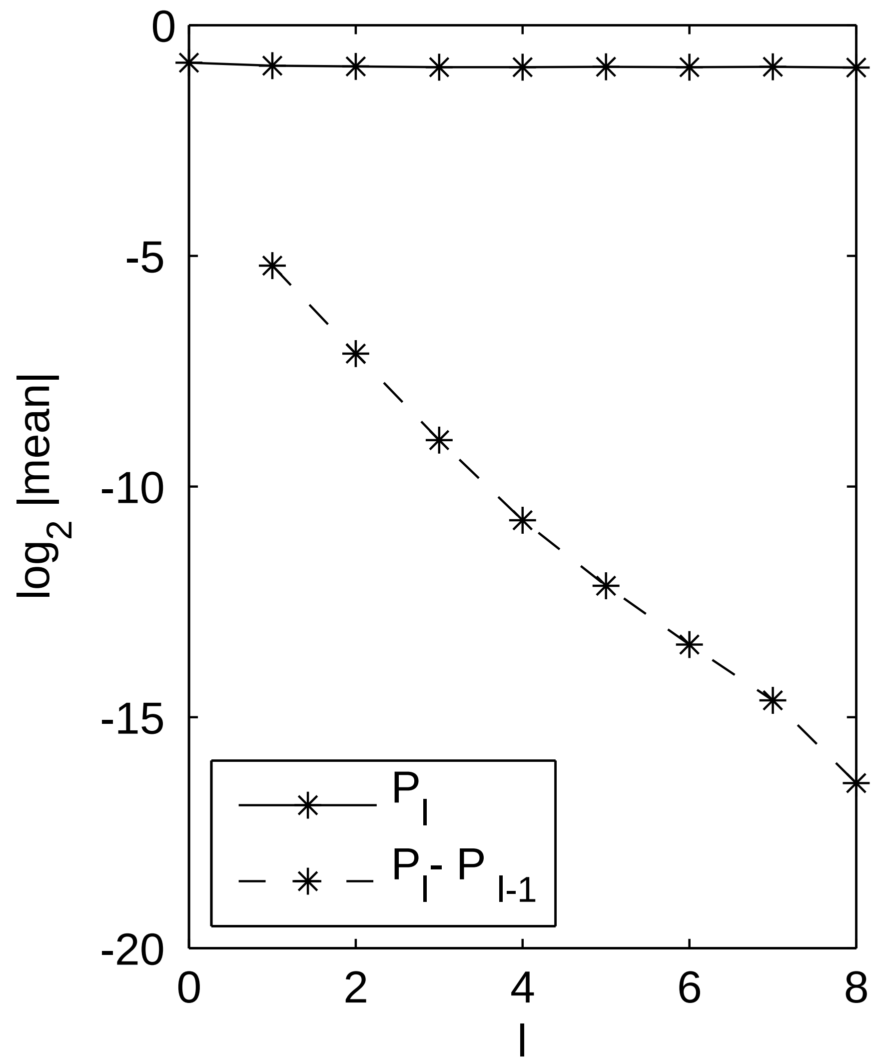
GBM: European call



MLMC Results

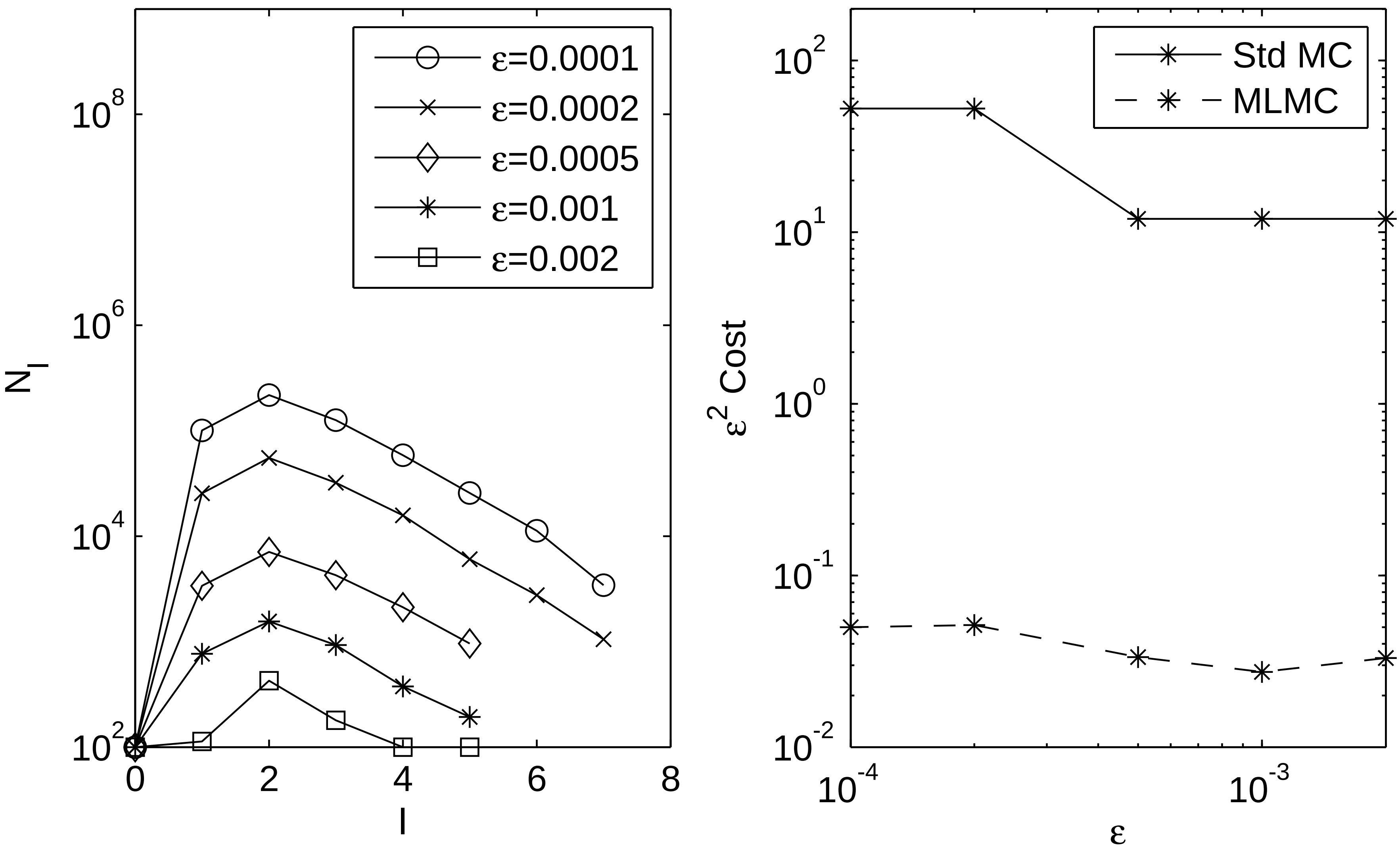
GBM: digital call





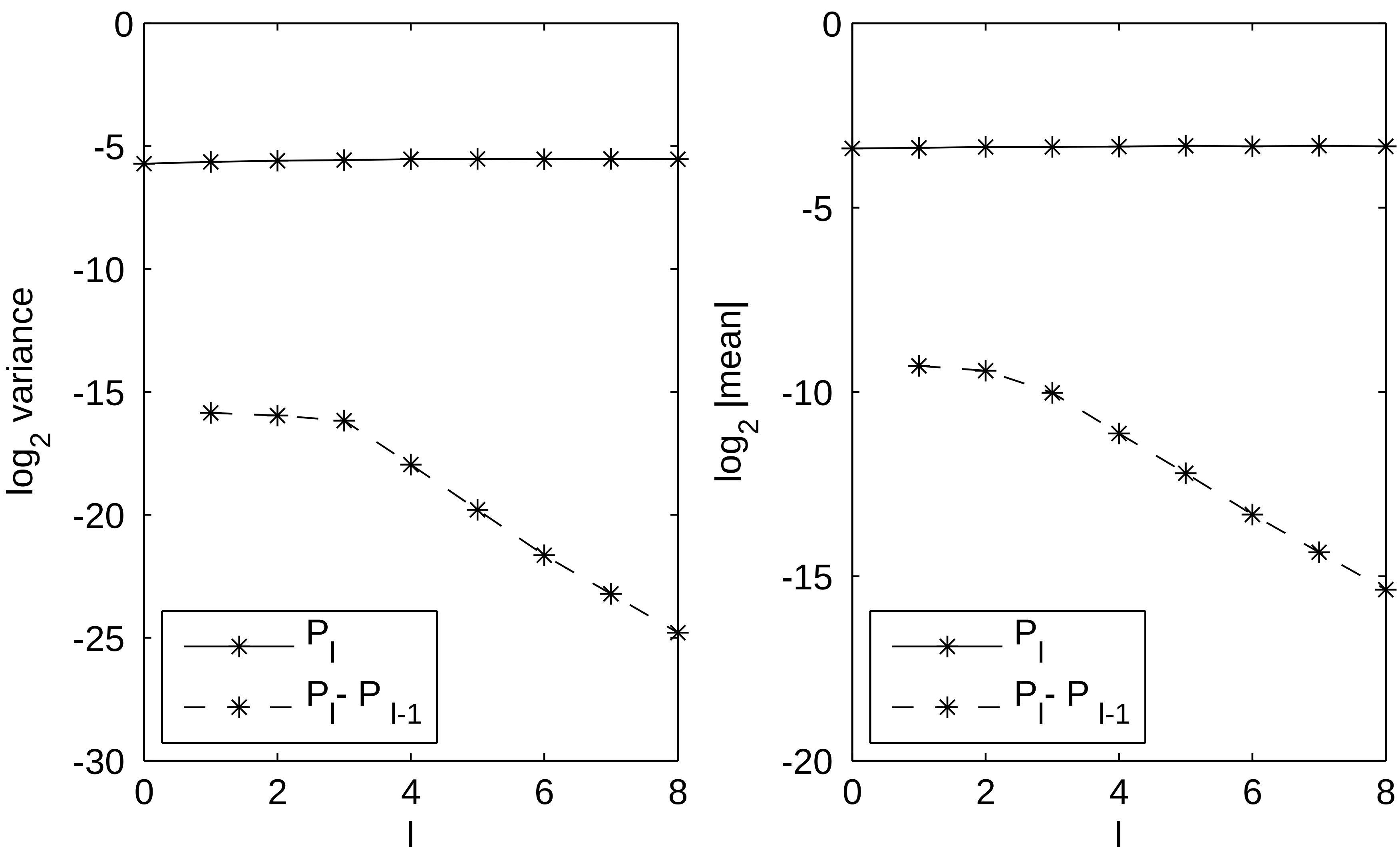
MLMC Results

GBM: digital call



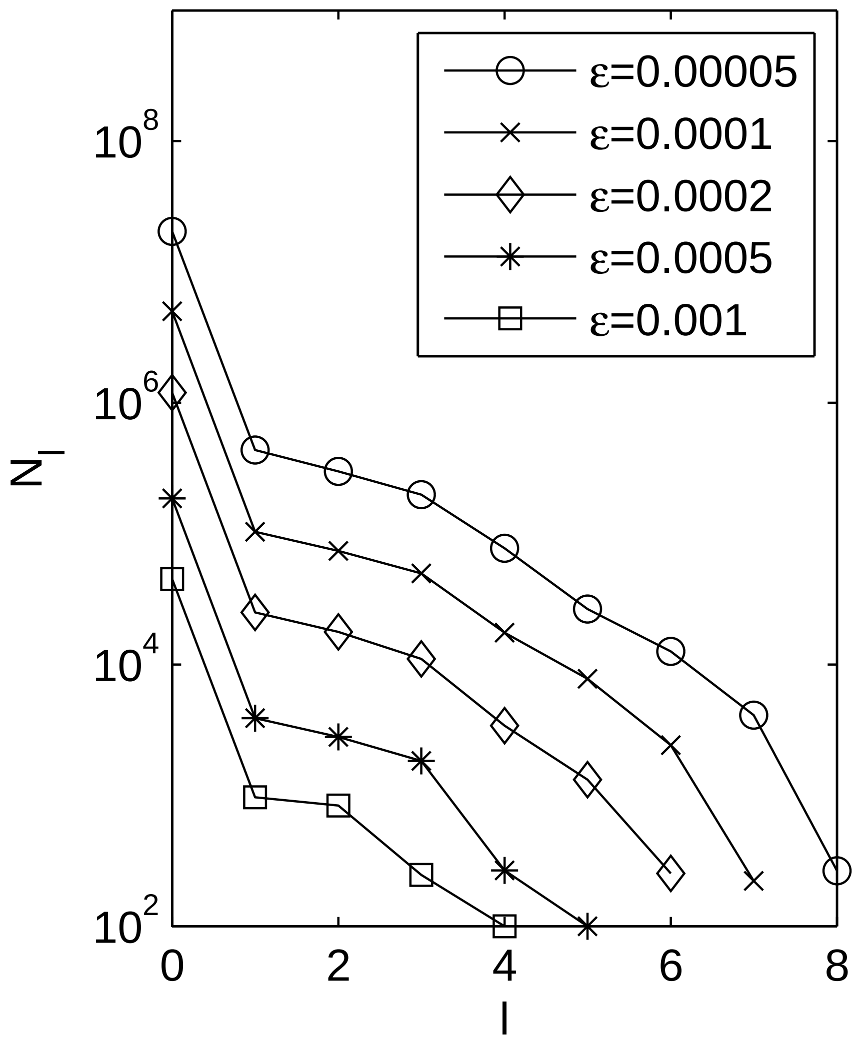
MLMC Results

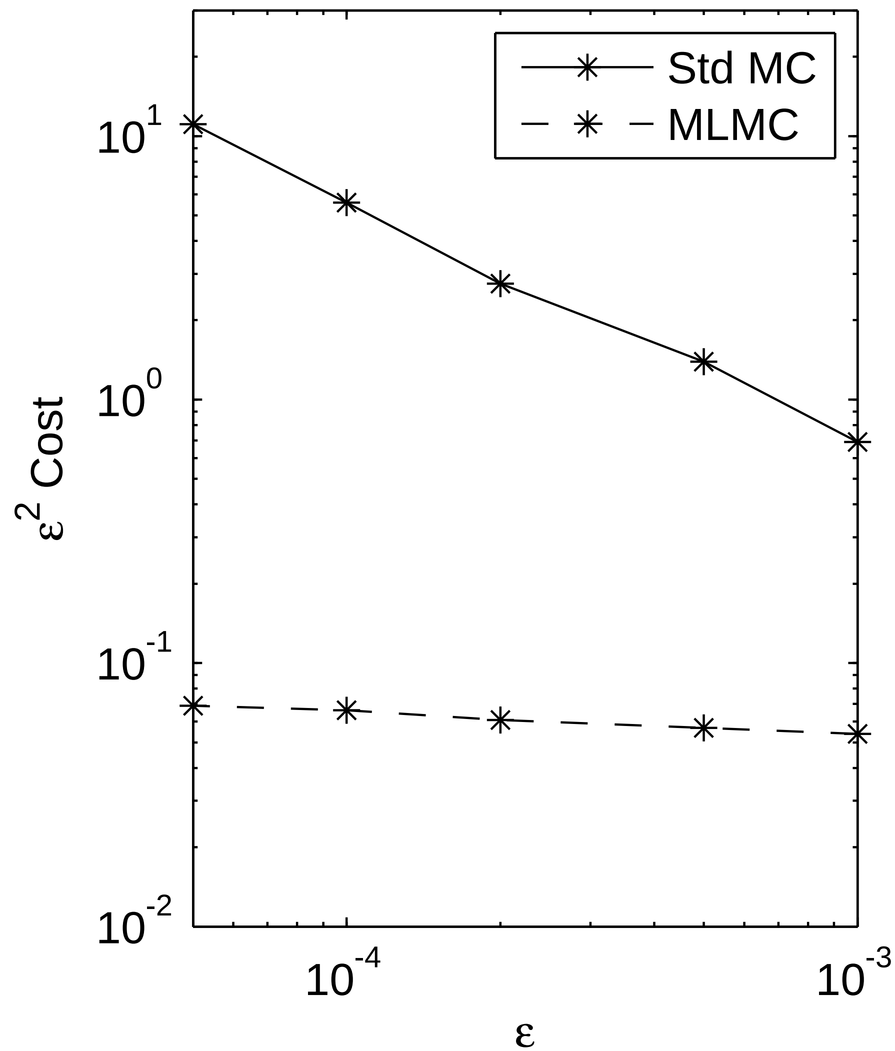
GBM: barrier option



MLMC Results

GBM: barrier option





Quasi-Monte Carlo

- well-established technique for approximating high-dimensional integrals
- for finance applications see papers by l'Ecuyer and book by Glasserman
- Sobol sequences are perhaps most popular; we used rank-1 lattice rules (Sloan & Kuo)
- two important ingredients for success:
  - randomized QMC for confidence intervals
  - good identification of "dominant dimensions" (Brownian Bridge and/or PCA)

Quasi-Monte Carlo

Approximate high-dimensional hypercube integral

$$\int_{[0,1]^d} f(x)\ \mathrm{d}x$$

by

$$\frac{1}{N}\sum_{i=0}^{N-1} f(x^{(i)})$$

where

$$x^{(i)} = \left[\frac{i}{N}\, z\right]$$

and z is a d-dimensional “generating vector”.

Quasi-Monte Carlo

In the best cases, error is $O(N^{-1})$ instead of $O(N^{-1/2})$ but without a confidence interval.

To get a confidence interval, let

$$x^{(i)} = \left[\frac{i}{N} \, z \; + \; x_0 \right].$$

where x_0 is a random offset vector.

Using 32 different random offsets gives a confidence interval in the usual way.

Quasi-Monte Carlo

For the path discretisation we can use

$$\Delta W_n = \sqrt{h}\, \Phi^{-1}(x_n),$$

where Φ^{-1} is the inverse cumulative Normal distribution.

Much better to use a Brownian Bridge construction:

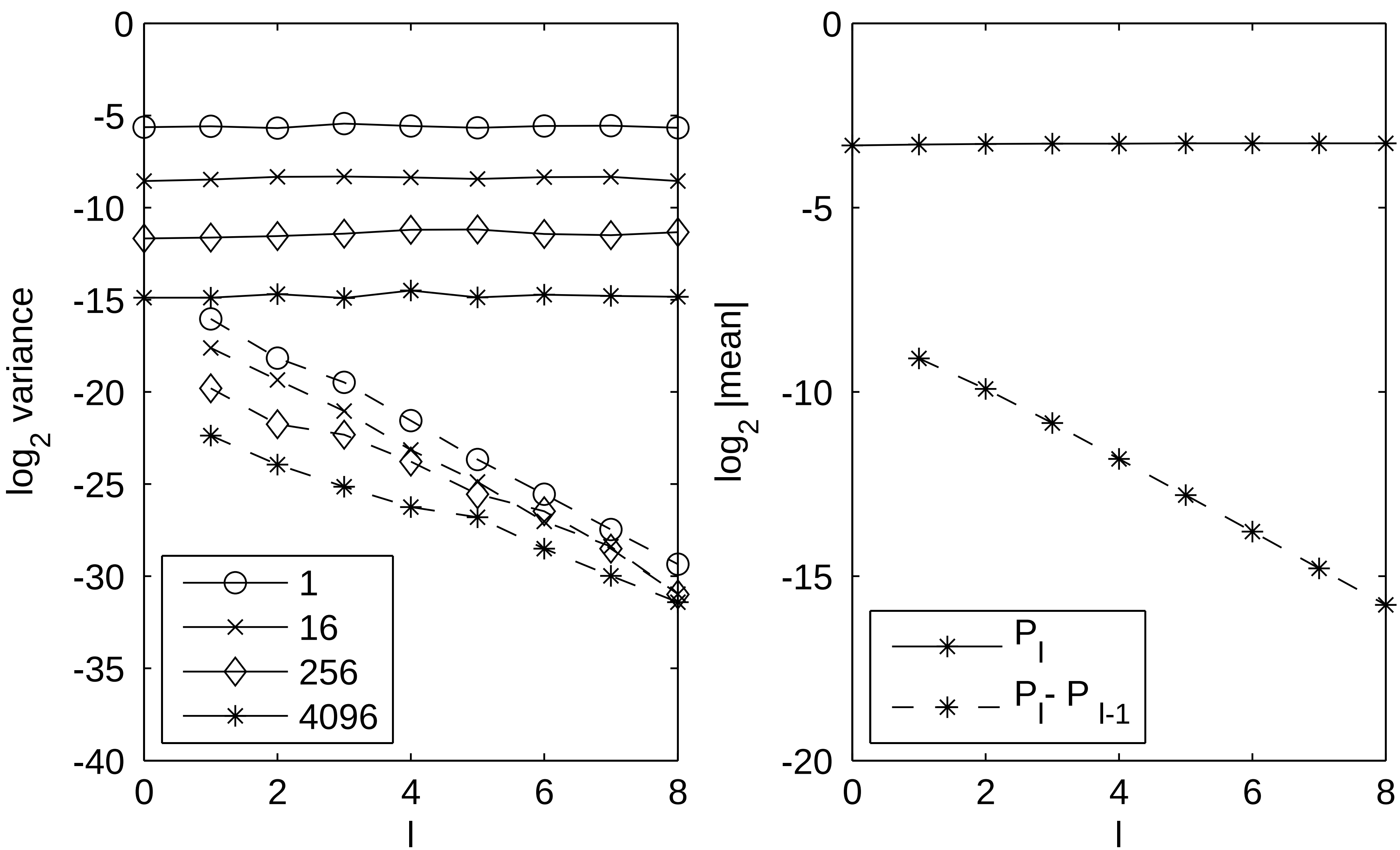
- $x_1 \quad \longrightarrow \quad W(T)$
- $x_2 \quad \longrightarrow \quad W(T/2)$
- $x_3, x_4 \quad \longrightarrow \quad W(T/4), W(3T/4)$
- … and so on by recursive bisection

Multilevel QMC

- rank-1 lattice rule developed by Sloan, Kuo & Waterhouse at UNSW
- 32 randomly-shifted sets of QMC points gives unbiased estimate and confidence interval for multilevel correction
- MLQMC algorithm uses same heuristic as MLMC algorithm to estimate weak error and choose optimal number of levels
- MLQMC algorithm repeatedly doubles the number of points on the level with greatest variance/cost ratio, until desired accuracy is achieved
- results show QMC to be particularly effective on lowest levels with low dimensionality

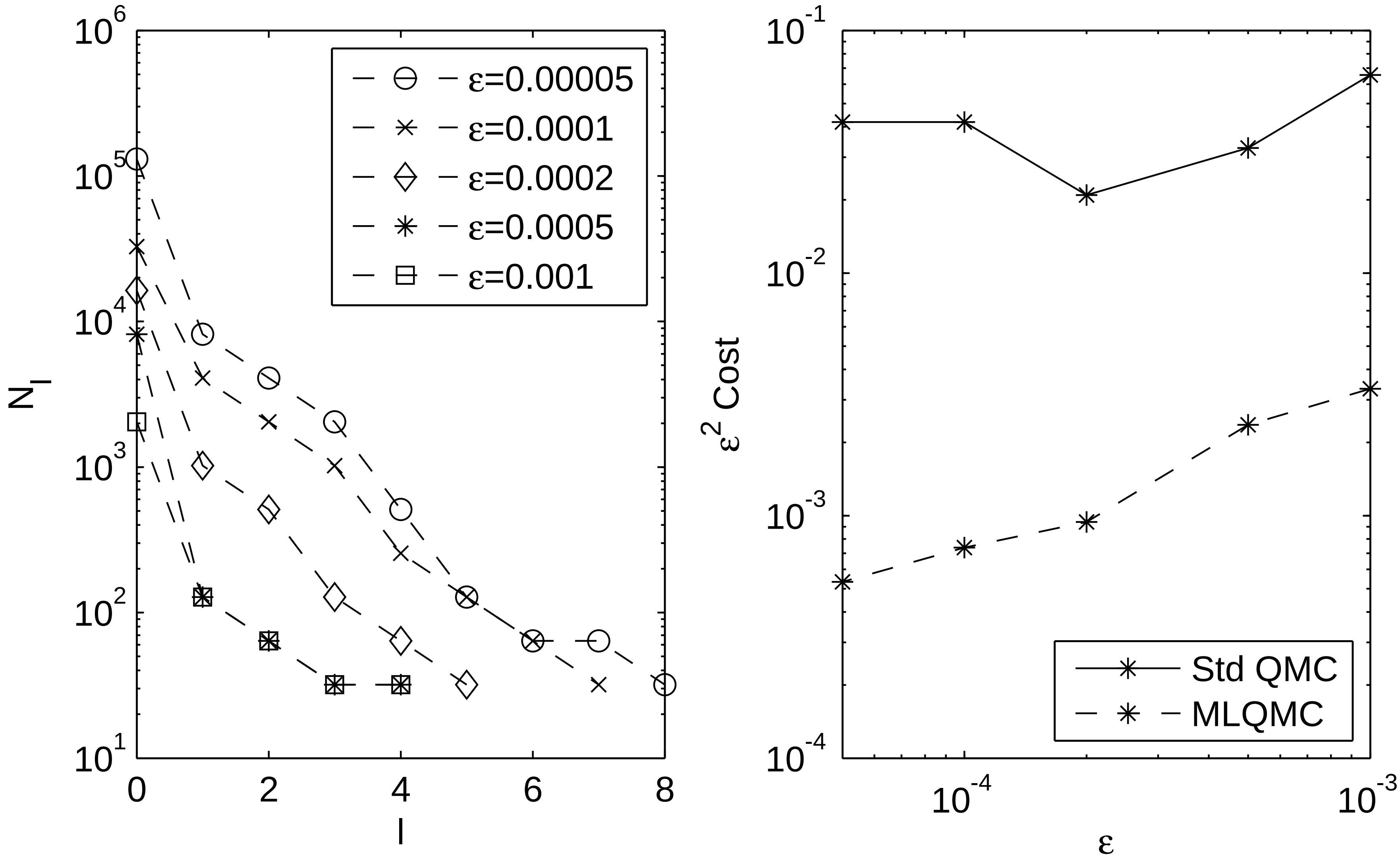
MLQMC Results

GBM: European call



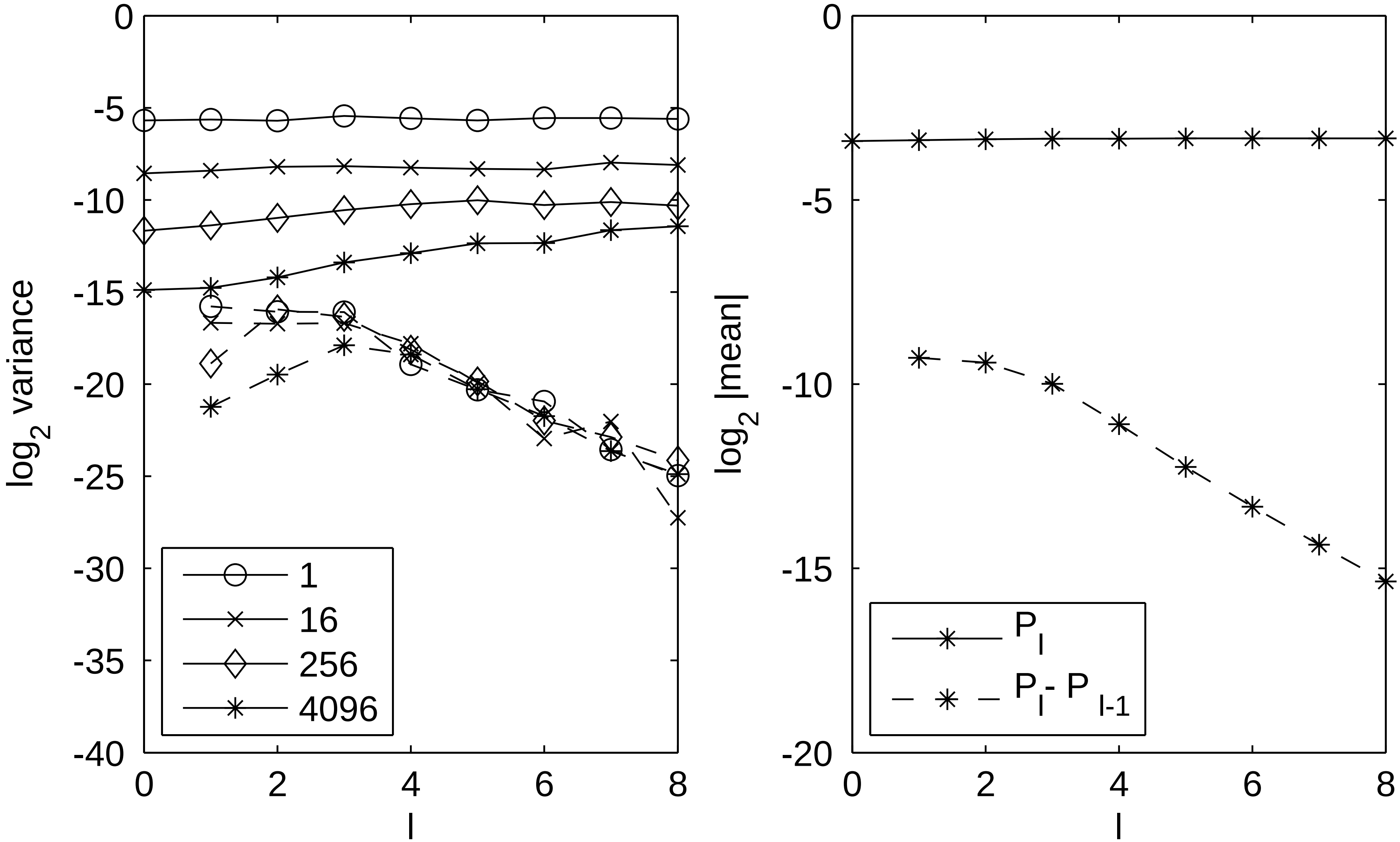
MLQMC Results

GBM: European call



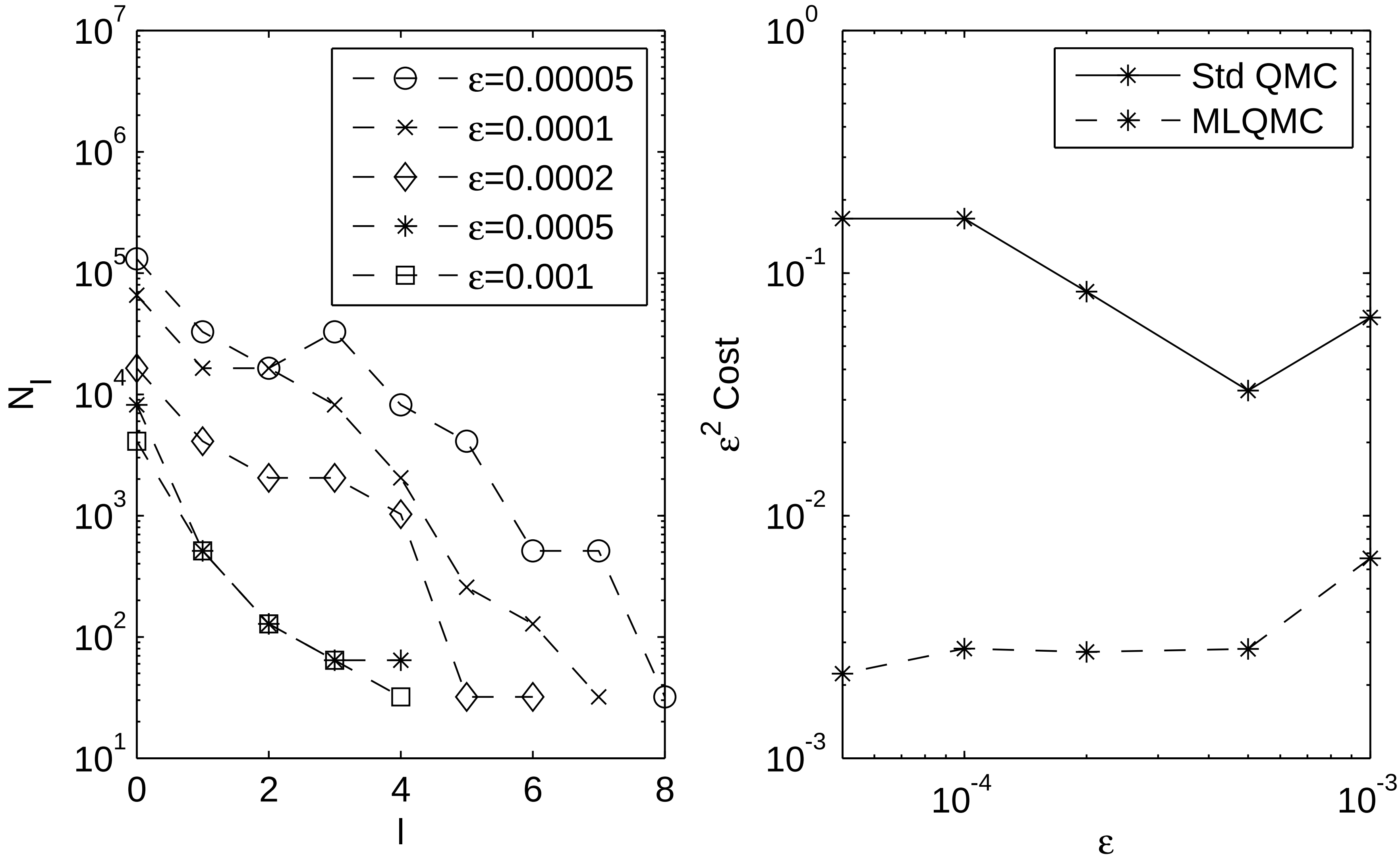
MLQMC Results

GBM: barrier option



MLQMC Results

GBM: barrier option



2015 postscript

This research (and almost all of the presentation) comes from 2007.

In May-June 2015, Adrien Grumberg repeated everything using Sobol points with Matousek–Owen digital scrambling.

The results were very similar – if anything, the Sobol results were slightly better.

A. Grumberg. “On Multilevel Quasi-Monte Carlo Methods”, MSc dissertation, Oxford, 2015.

people.maths.ox.ac.uk/gilesm/files/Adrien_Grumberg.pdf

Conclusions

- initial MLQMC research came very soon after MLMC
- numerical results were very encouraging, but there was no supporting numerical analysis
- Frances Kuo and Ian Sloan later developed MLQMC for SPDEs, in collaboration with Rob Scheichl, Christoph Schwab, and others – also made good progress on the numerical analysis

Future:

- more experiments
- new applications
  (e.g. continuous-time Markov processes)
- more numerical analysis

Papers

M.B. Giles, “Multilevel Monte Carlo path simulation”, *Operations Research*, 56(3):607-617, 2008.

M.B. Giles, “Improved multilevel convergence using the Milstein scheme”, pp.343-358 in *MCQMC06* proceedings, Springer-Verlag, 2008.

M.B. Giles, B. Waterhouse, “Multilevel quasi-Monte Carlo path simulation”, pp.165-181 in *Advanced Financial Modelling*, in Radon Series on Computational and Applied Mathematics, de Gruyter, 2009.

`people.maths.ox.ac.uk/gilesm/mlmc.html`

`people.maths.ox.ac.uk/gilesm/acta/`

`people.maths.ox.ac.uk/gilesm/mlmc_community.html`