Monte Carlo estimation of Greeks

(Smoking Adjoints: parts 2 and 3)

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“Smoking Adjoints”

Paper with Paul Glasserman in *Risk* in 2006 on the use of adjoints in computing pathwise sensitivities attracted a lot of interest, and questions:

- what is involved in practice in creating an adjoint code, and can it be simplified?
- do we really have to differentiate the payoff?
- what about non-differentiable payoffs?
- what about American options? (NOT addressed yet!)

Outline

- different approaches to computing Greeks
- adjoint pathwise sensitivities
- use of automatic differentiation
- “vibrato” Monte Carlo for non-differentiable payoffs

Generic Problem

Stochastic differential equation with general drift and volatility terms:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t)$$

For a simple European option we want to compute the expected discounted payoff value dependent on the terminal state:

$$V = \mathbb{E}[f(S(T))]$$

Note: the drift and volatility functions are almost always differentiable, but the payoff $f(S)$ is often not.

Generic Problem

Euler discretisation with timestep h:

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n$$

Simplest Monte Carlo estimator for expected payoff is an average of M independent path simulations:

$$M^{-1} \sum_{i=1}^{M} f(\widehat{S}^{(i)}_{T/h})$$

Greeks: for hedging and risk management we also want to estimate derivatives of expected payoff V

Simple Problem

For Geometric Brownian motion

$$\mathrm{d}S(t) = r\,S\,\mathrm{d}t + \sigma\,S\,\mathrm{d}W(t)$$

the SDE can be solved analytically to give

$$S(T) = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)\,T + \sigma\,W(T)\right)$$

In this case, we can directly sample $W(T)$ to get

$$V \equiv \mathbb{E}\left[f(S(T))\right] \approx M^{-1} \sum_{i=1}^{M} f(S^{(i)})$$

– will use this to explain approaches to calculating sensitivities

Evaluating sensitivities

Simplest approach is to use a finite difference approximation,

$$\frac{\partial V}{\partial \theta} \approx \frac{V(\theta+\Delta\theta) - V(\theta-\Delta\theta)}{2\,\Delta\theta}$$

$$\frac{\partial^2 V}{\partial \theta^2} \approx \frac{V(\theta+\Delta\theta) - 2\,V(\theta) + V(\theta-\Delta\theta)}{(\Delta\theta)^2}$$

– very simple, but expensive and inaccurate if $\Delta\theta$ is too big or too small

Evaluating sensitivities

For simple cases where we know the terminal probability distribution and so

$$V \equiv \mathbb{E}\left[f(S(T))\right] = \int f(S)\, p_S(S)\, \mathrm{d}S$$

we can differentiate this to get

$$\frac{\partial V}{\partial \theta} = \int f \frac{\partial p_S}{\partial \theta}\, \mathrm{d}S = \int f \frac{\partial (\log p_S)}{\partial \theta}\, p_S\, \mathrm{d}S = \mathbb{E}\left[f \frac{\partial (\log p_S)}{\partial \theta}\right]$$

This is the Likelihood Ratio Method – can handle non-differentiable payoffs, but doesn't generalise well to cases where we have to simulate the whole path

Evaluating sensitivities

Alternatively, for simple Geometric Brownian Motion

$$V \equiv \mathbb{E}\left[f(S(T))\right] = \int f(S(T))\, p_W(W)\, \mathrm{d}W$$

and differentiating this gives

$$\frac{\partial V}{\partial \theta} = \int \frac{\partial f}{\partial S}\, \frac{\partial S(T)}{\partial \theta}\, p_W\, \mathrm{d}W = \mathbb{E}\left[\frac{\partial f}{\partial S}\, \frac{\partial S(T)}{\partial \theta}\right]$$

with $\partial S(T)/\partial\theta$ being evaluated at fixed W.

This is the pathwise sensitivity approach – it can't handle non-differentiable payoffs, but does generalise well to cases where we have to simulate the whole path

Evaluating sensitivities

The generalisation involves differentiating the Euler path discretisation,

$$\widehat{S}_{n+1} = \widehat{S}_n + a(\widehat{S}_n, t_n)\, h + b(\widehat{S}_n, t_n)\, \Delta W_n$$

holding fixed the Brownian increments, to get

$$\frac{\partial \widehat{S}_{n+1}}{\partial \theta} = \left(1 + \frac{\partial a}{\partial S}\, h + \frac{\partial b}{\partial S}\, \Delta W_n\right) \frac{\partial \widehat{S}_n}{\partial \theta} + \frac{\partial a}{\partial \theta}\, h + \frac{\partial b}{\partial \theta}\, \Delta W_n$$

leading to

$$\frac{\partial \widehat{V}}{\partial \theta} = M^{-1} \sum_{i=1}^{M} \frac{\partial f}{\partial S}(\widehat{S}_N^{(i)})\ \frac{\partial \widehat{S}_N^{(i)}}{\partial \theta}.$$

Adjoint sensitivities

The adjoint approach is an efficient implementation of pathwise sensitivities.

Consider a process in which a vector input α leads to a final state vector S which is used to compute a scalar payoff P

$$\alpha \quad \longrightarrow \quad S \quad \longrightarrow \quad P$$

Taking $\dot{\alpha}, \dot{S}, \dot{P}$ to be the derivatives w.r.t. j^{th} component of α, then

$$\dot{S} = \frac{\partial S}{\partial \alpha}\, \dot{\alpha}, \qquad \dot{P} = \frac{\partial P}{\partial S}\, \dot{S},$$

and hence

$$\dot{P} = \frac{\partial P}{\partial S}\, \frac{\partial S}{\partial \alpha}\, \dot{\alpha}.$$

Adjoint sensitivities

Alternatively, defining $\overline{\alpha}, \overline{S}, \overline{P}$ to be the derivatives of P with respect to α, S, P, then

$$\overline{\alpha} \stackrel{\text{def}}{=} \left(\frac{\partial P}{\partial \alpha}\right)^T = \left(\frac{\partial P}{\partial S}\frac{\partial S}{\partial \alpha}\right)^T = \left(\frac{\partial S}{\partial \alpha}\right)^T \overline{S},$$

and similarly

$$\overline{S} = \left(\frac{\partial P}{\partial S}\right)^T \overline{P},$$

giving

$$\overline{\alpha} = \left(\frac{\partial S}{\partial \alpha}\right)^T \left(\frac{\partial P}{\partial S}\right)^T \overline{P}.$$

Adjoint sensitivities

The two are mathematically equivalent, since

$$\dot{P} = \frac{\partial P}{\partial \alpha}\,\dot{\alpha} = \overline{\alpha}^T \dot{\alpha} = \overline{\alpha}_j$$

but the adjoint approach approach is much cheaper because a single calculation gives $\overline{\alpha}$, the sensitivity of P to each one of the elements of α.

- standard approach: cost proportional to number of Greeks
- adjoint approach: cost independent
- crossover point for cost: 4 – 6 Greeks?

Adjoint sensitivities

Note that the standard approach goes forward

$$\dot{\alpha} \quad \longrightarrow \quad \dot{S} \quad \longrightarrow \quad \dot{P}$$

while the adjoint approach does the reverse

$$\overline{\alpha} \quad \longleftarrow \quad \overline{S} \quad \longleftarrow \quad \overline{P}.$$

These correspond to the forward and reverse modes of AD (Automatic Differentiation). “Smoking Adjoints” paper extended this to multiple timesteps in the path calculation — instead, we’ll now extend it to the steps in a whole computer program.

Automatic Differentiation

A computer instruction creates an additional new value:

$$\mathbf{u}^n \;=\; \mathbf{f}^n(\mathbf{u}^{n-1}) \;\equiv\; \begin{pmatrix} \mathbf{u}^{n-1} \\ f_n(\mathbf{u}^{n-1}) \end{pmatrix},$$

A computer program is the composition of N such steps:

$$\mathbf{u}^N = \mathbf{f}^N \circ \mathbf{f}^{N-1} \circ \ldots \circ \mathbf{f}^2 \circ \mathbf{f}^1(\mathbf{u}^0).$$

Automatic Differentiation

In forward mode, differentiation w.r.t. one element of the input vector gives

$$\dot{\mathbf{u}}^n = D^n \, \dot{\mathbf{u}}^{n-1}, \quad D^n \equiv \begin{pmatrix} I^{n-1} \\ \partial f_n / \partial \mathbf{u}^{n-1} \end{pmatrix},$$

and hence

$$\dot{\mathbf{u}}^N = D^N \, D^{N-1} \ldots \, D^2 \, D^1 \, \dot{\mathbf{u}}^0$$

Automatic Differentiation

In reverse mode, we consider the sensitivity of one element of the output vector, to get

$$\left(\overline{\mathbf{u}}^{n-1}\right)^T \equiv \frac{\partial u_i^N}{\partial \mathbf{u}^{n-1}} = \frac{\partial u_i^N}{\partial \mathbf{u}^n} \frac{\partial \mathbf{u}^n}{\partial \mathbf{u}^{n-1}} = \left(\overline{\mathbf{u}}^n\right)^T D^n,$$

$$\Longrightarrow \quad \overline{\mathbf{u}}^{n-1} = \left(D^n\right)^T \overline{\mathbf{u}}^n.$$

and hence

$$\overline{\mathbf{u}}^0 = \left(D^1\right)^T \left(D^2\right)^T \ldots \left(D^{N-1}\right)^T \left(D^N\right)^T \overline{\mathbf{u}}^N.$$

Note: need to go forward through original calculation to compute/store the D^n, then go in reverse to compute $\overline{\mathbf{u}}^n$

Automatic Differentiation

At the level of a single instruction

$$c = f(a, b)$$

the forward mode is

$$\begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix}^{n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial a} & \frac{\partial f}{\partial b} \end{pmatrix} \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix}^{n-1}$$

and so the reverse mode is

$$\begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix}^{n-1} = \begin{pmatrix} 1 & 0 & \frac{\partial f}{\partial a} \\ 0 & 1 & \frac{\partial f}{\partial b} \end{pmatrix} \begin{pmatrix} \overline{a} \\ \overline{b} \\ \overline{c} \end{pmatrix}^{n}$$

Automatic Differentiation

This gives a prescriptive algorithm for reverse mode differentiation.

Again the reverse mode is much more efficient if we want the sensitivity of a single output to multiple inputs.

Key result is that the cost of the reverse mode is at worst a factor 4 greater than the cost of the original calculation, regardless of how many sensitivities are being computed!

The storage of the D^n is minor for SDEs – much more of a concern for PDEs. There are also extra complexities when solving implicit equations through a fixed point iteration.

Automatic Differentiation

Manual implementation of the forward/reverse mode algorithms is possible but tedious.

Fortunately, automated tools have been developed, following one of two approaches:

- operator overloading (ADOL-C, FADBAD++)
- source code transformation (Tapenade, TAF/TAC++, ADIFOR)

My personal experience is with Tapenade for Fortran, and FADBAD++ for C++. Both are easy to use, Tapenade is as efficient as hand-coded, FADBAD++ less so.

Operator overloading

- define new datatype and associated arithmetic operators
- very natural for forward mode, but also works for reverse mode

$$x + \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x+y \\ \dot{y} \end{pmatrix} \qquad \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x+y \\ \dot{x}+\dot{y} \end{pmatrix}$$

$$x * \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x*y \\ x*\dot{y} \end{pmatrix} \qquad \begin{pmatrix} x \\ \dot{x} \end{pmatrix} * \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x*y \\ \dot{x}*y + x*\dot{y} \end{pmatrix}$$

$$x / \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x/y \\ -(x/y^2)*\dot{y} \end{pmatrix} \qquad \begin{pmatrix} x \\ \dot{x} \end{pmatrix} / \begin{pmatrix} y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x/y \\ \dot{x}/y - (x/y^2)*\dot{y} \end{pmatrix}$$

Operator overloading

```
template <typename ADdouble>
void path_calc(const int N, const int Nmat, const double delta,
               ADdouble L[], const double lambda[], const double z[])
{
  int      i, n;
  double   sqez, lam, con1;
  ADdouble v, vrat;

  for(n=0; n<Nmat; n++) {
    sqez = sqrt(delta)*z[n];

    v = 0.0;
    for (i=n+1; i<N; i++) {
      lam  = lambda[i-n-1];
      con1 = delta*lam;
      v   += (con1*L[i])/(1.0+delta*L[i]);
      vrat = exp(con1*v + lam*(sqez-0.5*con1));
      L[i] = L[i]*vrat;
    }
  }
}
```

Source code transformation

- programmer supplies code which takes u as input and produces $v = f(u)$ as output
- in forward mode, AD tool generates new code which takes u and $\dot{u}$ as input, and produces v and $\dot{v}$ as output

$$\dot{v} = \left(\frac{\partial f}{\partial u}\right) \dot{u}$$

- in reverse mode, AD tool generates new code which takes u and $\overline{v}$ as input, and produces v and $\overline{u}$ as output

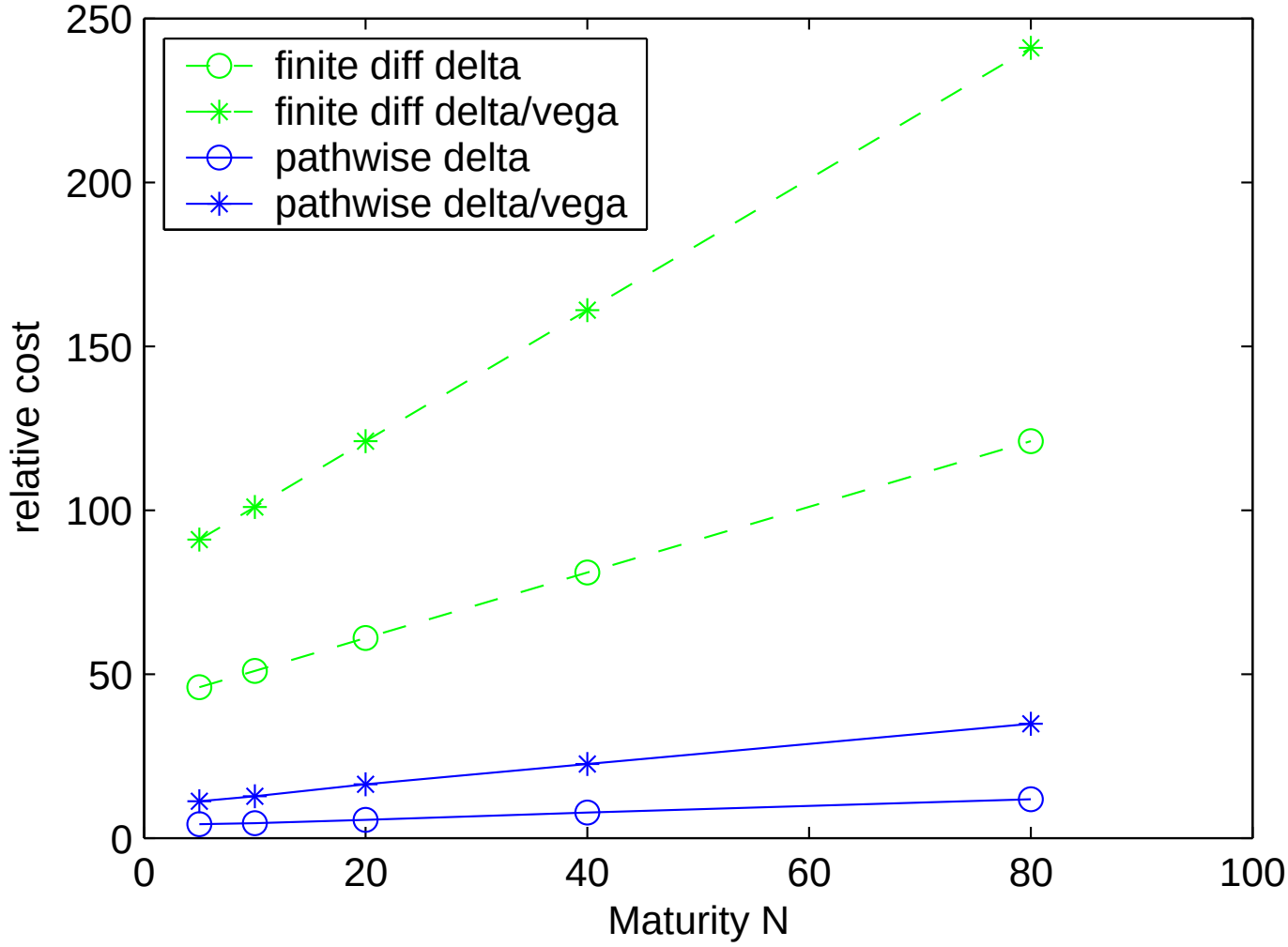
$$\overline{u} = \left(\frac{\partial f}{\partial u}\right)^T \overline{v}$$

LIBOR Application

- testcase from “Smoking Adjoints” paper
- test problem performs N timesteps with a vector of $N+40$ forward rates, and computes the $N+40$ deltas and vegas for a portfolio of swaptions
- originally hand-coded (using the ideas from AD), now used to test the effectiveness of AD tools

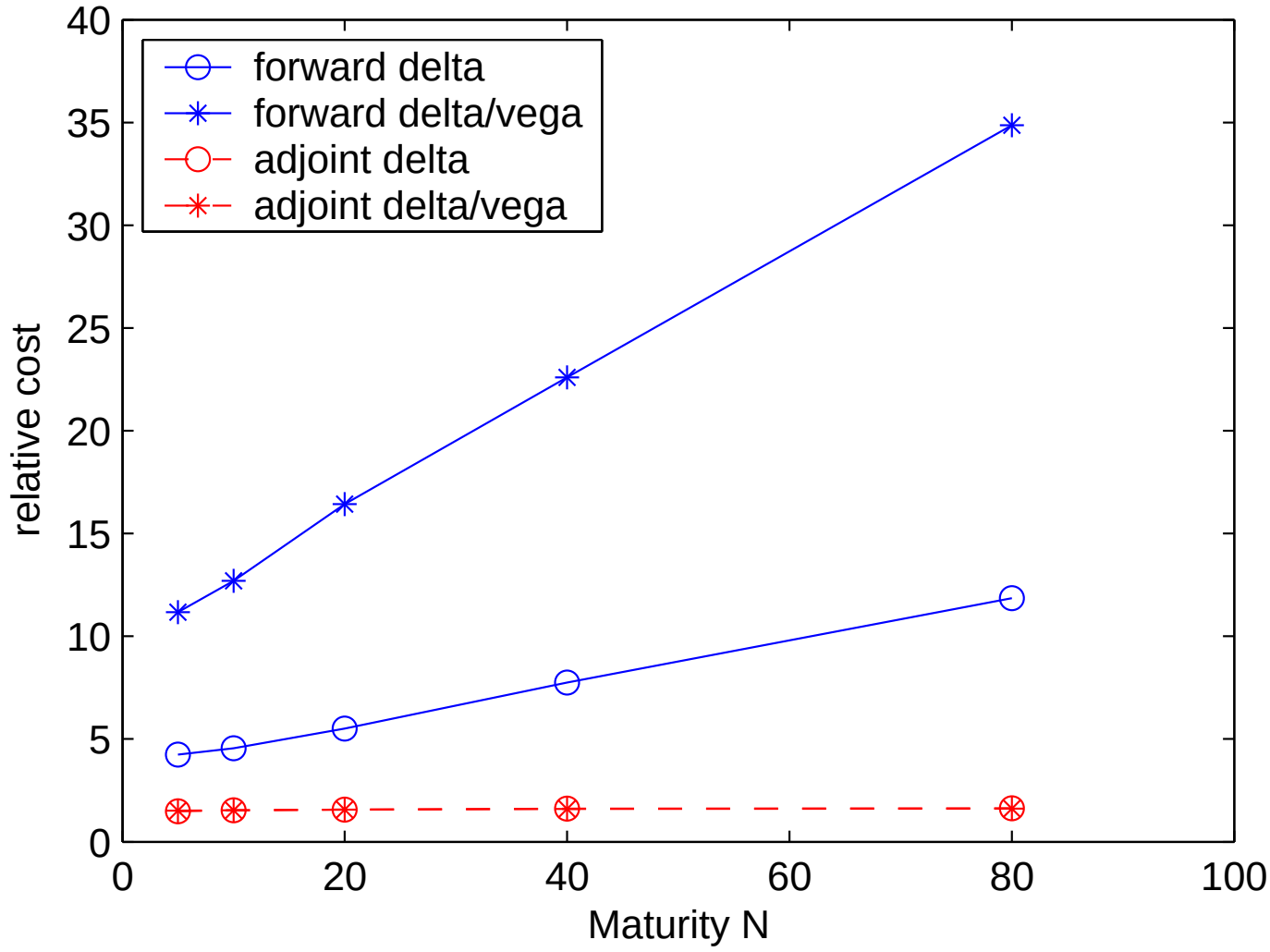
LIBOR Application

Finite differences versus forward pathwise sensitivities:



LIBOR Application

Hand-coded forward versus adjoint pathwise sensitivities:



LIBOR Application

Timings per path for $N = 40$; the hybrid version uses hand-coded for the path and FADBAD++ for the payoff

milliseconds/path	Gnu g++	Intel `icc`
original	0.37	0.10
hand-coded forward	0.97	0.52
hand-coded reverse	0.47	0.19
FADBAD++ forward	4.30	5.00
FADBAD++ reverse	6.20	4.86
hybrid forward	1.02	0.63
hybrid reverse	0.65	0.35
TAC++ forward	1.36	0.85
TAC++ reverse	1.28	0.45

Automatic Differentiation

Conclusions?

- hand-coded is clearly most efficient
  I optimised the implementation (tradeoff between storage versus recomputation) in a way the AD tools cannot yet.
- however, AD code is very useful for debugging/validating hand-coded version, and can be used for bits which are not computationally intensive
- AD is likely to be useful for bigger applications (vital in computational science and engineering)

Vibrato Monte Carlo

One remaining problem – what if payoff is not differentiable?

- LRM
  - estimator variance proportional to h^{-1}
- Malliavin calculus
  - recent paper by Glasserman and Chen shows it can be viewed as a pathwise/LRM hybrid
  - might be good choice when few Greeks needed
- new “vibrato” Monte Carlo idea
  - also a pathwise/LRM hybrid
  - variance proportional to $h^{-1/2}$
  - efficient adjoint implementation

Vibrato Monte Carlo

- new idea, based on Glasserman example of conditional expectation for a simple digital option
- output of each SDE path calculation becomes a narrow (multivariate) Normal distribution
- combine pathwise sensitivity for the differentiable SDE, with LRM for the non-differentiable payoff
- avoiding the differentiation of the payoff also simplifies the implementation in real-world setting

Vibrato Monte Carlo

Final timestep of Euler path discretisation is

$$\widehat{S}_N = \widehat{S}_{N-1} + a(\widehat{S}_{N-1}, t_{N-1})\, h + b(\widehat{S}_{N-1}, t_{N-1})\, \Delta W_{N-1}$$

Instead of using random number generator to get a value for ΔW_{N-1}, consider the whole distribution of possible values, so $\widehat{S}_N$ has a Normal distribution with mean

$$\mu(W) = \widehat{S}_{N-1} + a(\widehat{S}_{N-1}, t_{N-1})\, h$$

and standard deviation

$$\sigma(W) = b(\widehat{S}_{N-1}, t_{N-1})\, \sqrt{h}$$

where $W \equiv (\Delta W_0, \Delta W_1, \ldots \Delta W_{N-2})$.

Vibrato Monte Carlo

For a particular path given by a particular vector W, the expected payoff is

$$\mathbb{E}_Z[f(\mu+\sigma\, Z)]$$

where Z is a Normal random variable with zero mean and unit variance.

Averaging over all W then gives the same overall expectation as before.

Note also that, for given W, $\widehat{S}_N$ has a Normal distribution with

$$p_S(\widehat{S}_N) = \frac{1}{\sqrt{2\pi}\,\sigma}\ \exp\left(-\frac{(\widehat{S}_N-\mu)^2}{2\,\sigma^2}\right)$$

Vibrato Monte Carlo

In the case of a simple digital call with strike K, Glasserman uses the analytic solution

$$\mathbb{E}_Z[f(\mu+\sigma\, Z)] = \exp(-rT)\ \Phi\left(\frac{\mu-K}{\sigma}\right).$$

- for each W, the payoff is now smooth, differentiable
- derivative is $O(h^{-1/2})$ near strike, near zero elsewhere
  $\Longrightarrow$ variance is $O(h^{-1/2})$
- analytic evaluation of conditional expectation not possible in general for multivariate cases
  $\Longrightarrow$ use Monte Carlo estimation!

Vibrato Monte Carlo

Main novelty comes in calculating the sensitivity.

For a particular W, we have a Normal probability distribution for $\widehat{S}_N$ and can apply the Likelihood Ratio method to get

$$\frac{\partial}{\partial\theta}\,\mathbb{E}_Z\left[f(\widehat{S}_N)\right] \;=\; \mathbb{E}_Z\left[f(\widehat{S}_N)\,\frac{\partial(\log p_S)}{\partial\theta}\right],$$

where

$$\begin{aligned}\frac{\partial(\log p_S)}{\partial\theta} &= \frac{\partial(\log p_S)}{\partial\mu}\,\frac{\partial\mu}{\partial\theta} + \frac{\partial(\log p_S)}{\partial\sigma}\,\frac{\partial\sigma}{\partial\theta}\\ &= \frac{Z}{\sigma}\,\frac{\partial\mu}{\partial\theta} + \frac{Z^2-1}{\sigma}\,\frac{\partial\sigma}{\partial\theta}.\end{aligned}$$

Averaging over all W then gives the expected sensitivity.

Vibrato Monte Carlo

To improve the variance, we note that

$$\mathbb{E}[1] = 1 \quad \Longrightarrow \quad \mathbb{E}_Z\left[\frac{\partial(\log p_S)}{\partial\theta}\right] = 0$$

and hence

$$\frac{\partial}{\partial\theta}\,\mathbb{E}_Z\left[f(\widehat{S}_N)\right] \;=\; \mathbb{E}_Z\left[\Big(f(\mu+\sigma Z)-f(\mu)\Big)\,\frac{\partial(\log p_S)}{\partial\theta}\right].$$

The quantity

$$\widehat{P} = \Big(f(\mu+\sigma Z)-f(\mu)\Big)\,\frac{\partial(\log p_S)}{\partial\theta}$$

has $O(1)$ variance when $f(S)$ is Lipschitz.

Vibrato Monte Carlo

In the multivariate extension with mean vector μ and covariance matrix Σ,

$$\widehat{S}(W,Z) = \mu + C\,Z$$

where $\Sigma = C\,C^T$ and Z is a vector of uncorrelated Normals. The joint p.d.f. is

$$\log p_S = -\tfrac{1}{2}\log|\Sigma| - \tfrac{1}{2}(\widehat{S}-\mu)^T\Sigma^{-1}(\widehat{S}-\mu) - \tfrac{1}{2}d\,\log(2\pi).$$

and so

$$\frac{\partial \log p_S}{\partial \mu} = C^{-T}Z,$$

$$\frac{\partial \log p_S}{\partial \Sigma} = \tfrac{1}{2}\,C^{-T}\left(ZZ^T - I\right)C^{-1}$$

Vibrato Monte Carlo

For each W, in forward mode we have

$$\alpha,\ \dot{\alpha} \quad \longrightarrow \quad \mu,\ \dot{\mu},\ \Sigma,\ \dot{\Sigma} \quad \longrightarrow \quad \text{payoff + sensitivity}$$

- first bit – pathwise sensitivity calculation
- second bit – Likelihood Ratio Method

For maximum efficiency can use adjoint/reverse mode

$$\alpha \quad \longrightarrow \quad \mu,\ \Sigma \quad \longrightarrow \quad \text{payoff}$$

$$\overline{\alpha} \quad \longleftarrow \quad \overline{\mu},\ \overline{\Sigma} \quad \longleftarrow \quad \text{sensitivity}$$

$\overline{\mu},\ \overline{\Sigma}$ are coefficients multiplying $\dot{\mu},\ \dot{\Sigma}$ in forward mode

Conclusions

Monte Carlo estimation of Greeks is an important problem in computational finance

Improved methods need ideas from both mathematics

- adjoint technique
- vibrato Monte Carlo

. . . and computer science

- automatic differentiation

Future Work

- application of AD to real-world problems
- implementation/demonstration of vibrato MC
- combining these with multilevel MC (or QMC) for greater savings
- more consideration of pros and cons of Malliavin approach
- parallel execution on NVIDIA graphics cards

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Further information

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