Multilevel Monte Carlo methods

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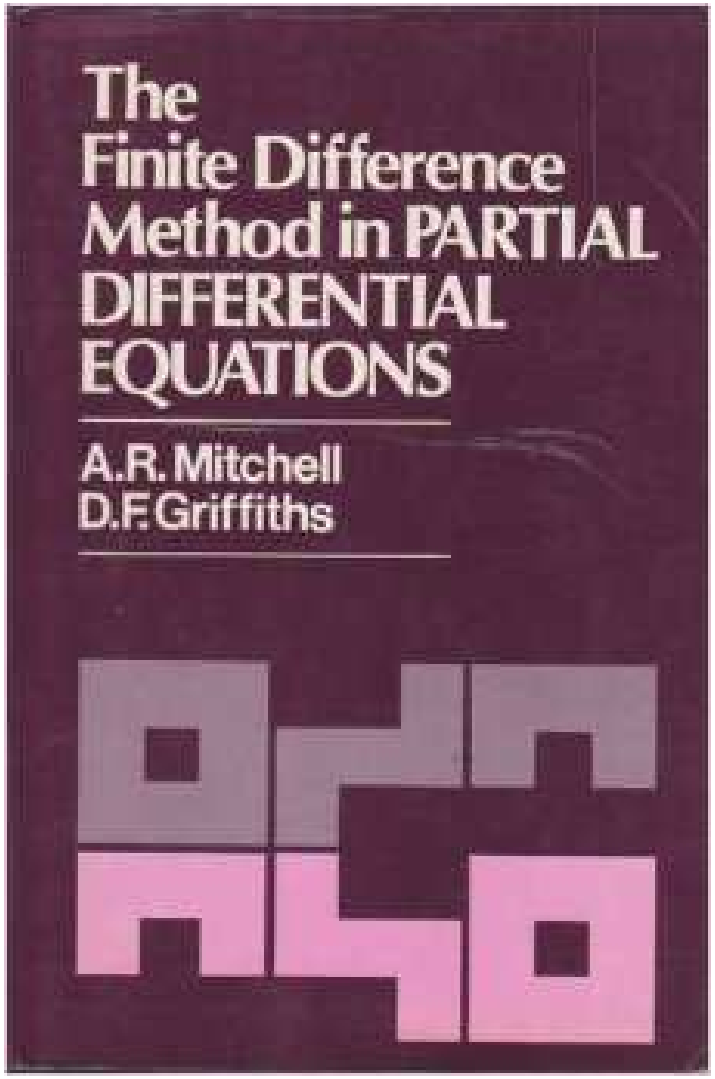
An outline history

- inspired by undergraduate numerical projects course at Cambridge, and summer projects at Rolls-Royce
- this was one of my first textbooks
- after 25 years working on CFD, 10 years ago I switched to Monte Carlo methods for computational finance and other application areas



Objectives

In presenting the multilevel Monte Carlo method, I hope to emphasise:

- the simplicity of the idea
- its flexibility – it's not prescriptive, more an approach
- the scope for improved performance through being creative
- there are lots of people working on a variety of applications

In doing this, I will focus on ideas rather than lots of numerical results.

Monte Carlo method

In stochastic models, we often have

$$\underset{\text{random input}}{\omega} \longrightarrow \underset{\text{intermediate variables}}{S} \longrightarrow \underset{\text{scalar output}}{P}$$

The Monte Carlo estimate for $\mathbb{E}[P]$ is an average of N independent samples $\omega^{(n)}$:

$$Y = N^{-1} \sum_{n=1}^{N} P(\omega^{(n)}).$$

This is unbiased, $\mathbb{E}[Y] = \mathbb{E}[P]$, and the Central Limit Theorem proves that as $N \to \infty$ the error becomes Normally distributed with variance $N^{-1}\mathbb{V}[P]$.

Monte Carlo method

In many cases, this is modified to

$$\underset{\text{random input}}{\omega} \longrightarrow \underset{\text{intermediate variables}}{\widehat{S}} \longrightarrow \underset{\text{scalar output}}{\widehat{P}}$$

where $\widehat{S}, \widehat{P}$ are approximations to S, P, in which case the MC estimate

$$\widehat{Y} = N^{-1} \sum_{n=1}^{N} \widehat{P}(\omega^{(n)})$$

is biased, and the Mean Square Error is

$$\mathbb{E}[\,(\widehat{Y} - \mathbb{E}[P])^2] \;=\; N^{-1}\,\mathbb{V}[\widehat{P}] + \Big(\mathbb{E}[\widehat{P}] - \mathbb{E}[P]\Big)^2$$

Greater accuracy requires larger N and smaller weak error $\mathbb{E}[\widehat{P}] - \mathbb{E}[P]$.

SDE Path Simulation

My interest was in SDEs (stochastic differential equations) for finance, which in a simple one-dimensional case has the form

$$\mathrm{d}S_t = a(S_t, t)\,\mathrm{d}t + b(S_t, t)\,\mathrm{d}W_t$$

Here $\mathrm{d}W_t$ is the increment of a Brownian motion – Normally distributed with variance $\mathrm{d}t$.

This is usually approximated by the simple Euler-Maruyama method

$$\widehat{S}_{t_{n+1}} = \widehat{S}_{t_n} + a(\widehat{S}_{t_n}, t_n)\,h + b(\widehat{S}_{t_n}, t_n)\,\Delta W_n$$

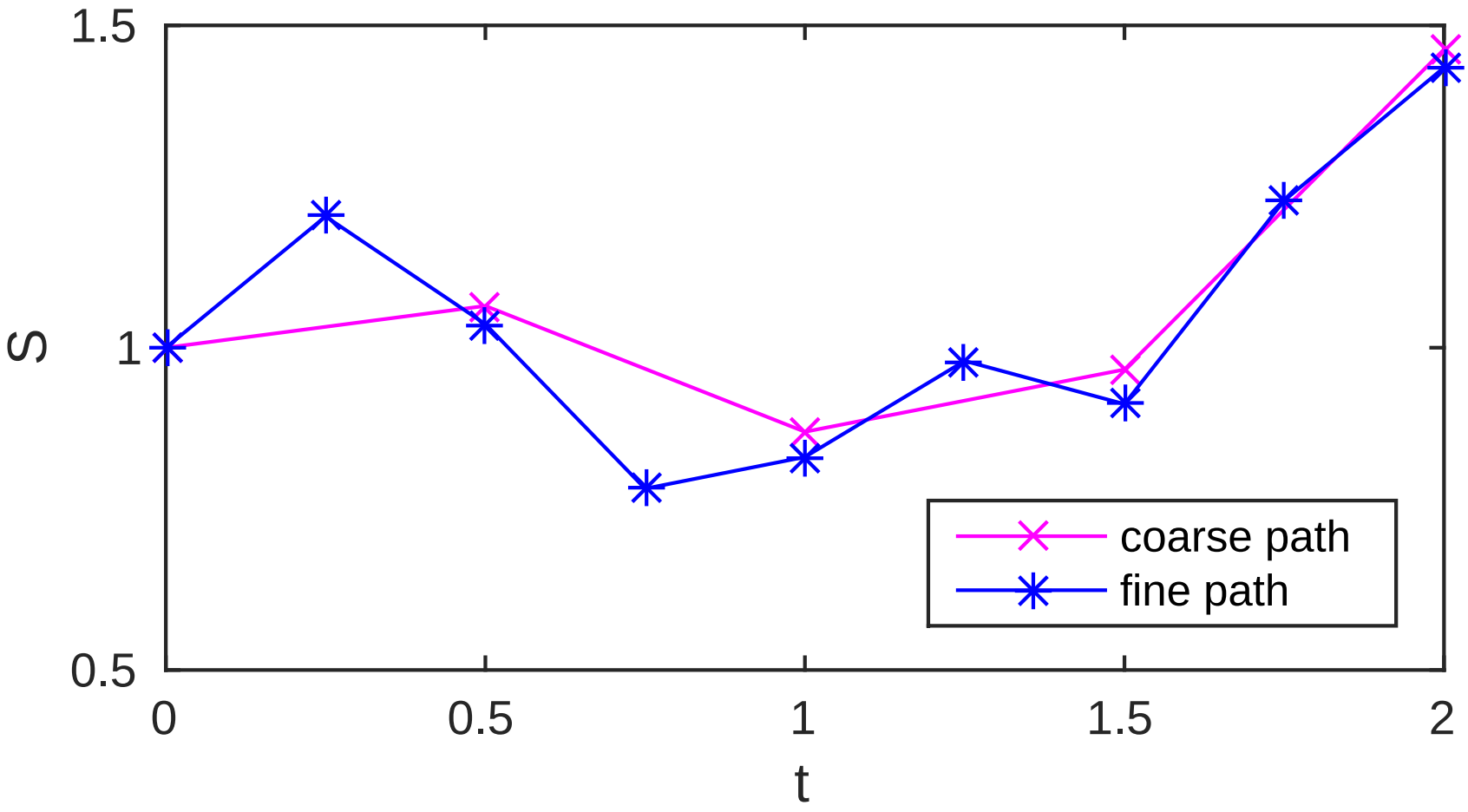
with uniform timestep h, and increments ΔW_n with variance h.

In simple applications, the output of interest is a function of the final value:

$$\widehat{P} \equiv f(\widehat{S}_T)$$

SDE Path Simulation

Geometric Brownian Motion: $\mathrm{d}S_t = r\,S_t\,\mathrm{d}t + \sigma\,S_t\,\mathrm{d}W_t$



SDE Path Simulation

Two kinds of discretisation error:

Weak error:

$$\mathbb{E}[\widehat{P}] - \mathbb{E}[P] = O(h)$$

Strong error:

$$\left(\mathbb{E}\left[\sup_{[0,T]}\left(\widehat{S}_t - S_t\right)^2\right]\right)^{1/2} = O(h^{1/2})$$

For reasons which will become clear, I prefer to use the Milstein discretisation for which the weak and strong errors are both $O(h)$.

SDE Path Simulation

The Mean Square Error is

$$N^{-1}\,\mathbb{V}[\widehat{P}] + \Big(\mathbb{E}[\widehat{P}] - \mathbb{E}[P]\Big)^2 \;\approx\; a\,N^{-1} + b\,h^2$$

If we want this to be ε^2, then we need

$$N = O(\varepsilon^{-2}), \qquad h = O(\varepsilon)$$

so the total computational cost is $O(\varepsilon^{-3})$.

To improve this cost we need to

- reduce N – variance reduction or Quasi-Monte Carlo methods
- reduce the cost of each path (on average) – MLMC

Two-level Monte Carlo

If we want to estimate $\mathbb{E}[\widehat{P}_1]$ but it is much cheaper to simulate $\widehat{P}_0 \approx \widehat{P}_1$, then since

$$\mathbb{E}[\widehat{P}_1] = \mathbb{E}[\widehat{P}_0] + \mathbb{E}[\widehat{P}_1 - \widehat{P}_0]$$

we can use the estimator

$$N_0^{-1} \sum_{n=1}^{N_0} \widehat{P}_0^{(0,n)} \;+\; N_1^{-1} \sum_{n=1}^{N_1} \left(\widehat{P}_1^{(1,n)} - \widehat{P}_0^{(1,n)} \right)$$

Benefit: if $\widehat{P}_1 - \widehat{P}_0$ is small, its variance will be small, so won't need many samples to accurately estimate $\mathbb{E}[\widehat{P}_1 - \widehat{P}_0]$, so cost will be reduced greatly.

Multilevel Monte Carlo

Natural generalisation: given a sequence $\widehat{P}_0, \widehat{P}_1, \ldots, \widehat{P}_L$

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{\ell=1}^{L} \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$$

we can use the estimator

$$N_0^{-1} \sum_{n=1}^{N_0} \widehat{P}_0^{(0,n)} \; + \; \sum_{\ell=1}^{L} \left\{ N_\ell^{-1} \sum_{n=1}^{N_\ell} \left(\widehat{P}_\ell^{(\ell,n)} - \widehat{P}_{\ell-1}^{(\ell,n)} \right) \right\}$$

with independent estimation for each level of correction

Multilevel Monte Carlo

If we define

- C_0, V_0 to be cost and variance of $\widehat{P}_0$
- C_ℓ, V_ℓ to be cost and variance of $\widehat{P}_\ell - \widehat{P}_{\ell-1}$

then the total cost is $\displaystyle\sum_{\ell=0}^{L} N_\ell\, C_\ell$ and the variance is $\displaystyle\sum_{\ell=0}^{L} N_\ell^{-1} V_\ell$.

Using a Lagrange multiplier μ^2 to minimise the cost for a fixed variance

$$\frac{\partial}{\partial N_\ell} \sum_{k=0}^{L} \left(N_k\, C_k + \mu^2 N_k^{-1} V_k \right) = 0$$

gives

$$N_\ell = \mu \sqrt{V_\ell / C_\ell} \quad \Longrightarrow \quad N_\ell\, C_\ell = \mu \sqrt{V_\ell\, C_\ell}$$

Multilevel Monte Carlo

Setting the total variance equal to ε^2 gives

$$\mu = \varepsilon^{-2} \left(\sum_{\ell=0}^{L} \sqrt{V_\ell \, C_\ell} \right)$$

and hence, the total cost is

$$\sum_{\ell=0}^{L} N_\ell \; C_\ell \;=\; \varepsilon^{-2} \left(\sum_{\ell=0}^{L} \sqrt{V_\ell C_\ell} \right)^2$$

in contrast to the standard cost which is approximately $\varepsilon^{-2} \, V_0 \, C_L$.

The MLMC cost savings are therefore approximately:

- V_L/V_0, if $\sqrt{V_\ell C_\ell}$ increases with level
- C_0/C_L, if $\sqrt{V_\ell C_\ell}$ decreases with level

Multilevel Path Simulation

With SDEs, level ℓ corresponds to approximation using M^ℓ timesteps, giving approximate payoff $\widehat{P}_\ell$ at cost $C_\ell = O(h_\ell^{-1})$.

Simplest estimator for $\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}]$ for $\ell > 0$ is

$$\widehat{Y}_\ell = N_\ell^{-1} \sum_{n=1}^{N_\ell} \left(\widehat{P}_\ell^{(n)} - \widehat{P}_{\ell-1}^{(n)} \right)$$

using same driving Brownian path for both levels.

Analysis gives MSE $= \displaystyle\sum_{\ell=0}^{L} N_\ell^{-1} V_\ell + \Big(\mathbb{E}[\widehat{P}_L] - \mathbb{E}[P] \Big)^2$

To make RMS error less than ε

- choose $N_\ell \propto \sqrt{V_\ell / C_\ell}$ so total variance is less than $\frac{1}{2}\,\varepsilon^2$
- choose L so that $\Big(\mathbb{E}[\widehat{P}_L] - \mathbb{E}[P] \Big)^2 < \frac{1}{2}\,\varepsilon^2$

Multilevel Path Simulation

For Lipschitz payoff functions $P \equiv f(S_T)$, we have

$$\begin{aligned} V_\ell \;\equiv\; \mathbb{V}\left[\widehat{P}_\ell - \widehat{P}_{\ell-1}\right] \;&\leq\; \mathbb{E}\left[(\widehat{P}_\ell - \widehat{P}_{\ell-1})^2\right] \\ &\leq\; K^2\,\mathbb{E}\left[(\widehat{S}_{T,\ell} - \widehat{S}_{T,\ell-1})^2\right] \\ &=\; \begin{cases} O(h_\ell), & \text{Euler-Maruyama} \\ O(h_\ell^2), & \text{Milstein} \end{cases} \end{aligned}$$

and hence

$$V_\ell\, C_\ell = \begin{cases} O(1), & \text{Euler-Maruyama} \\ O(h_\ell), & \text{Milstein} \end{cases}$$

MLMC Theorem

(Slight generalisation of version in 2008 *Operations Research* paper)

If there exist independent estimators $\widehat{Y}_\ell$ based on N_ℓ Monte Carlo samples, each costing C_ℓ, and positive constants $\alpha, \beta, \gamma, c_1, c_2, c_3$ such that $\alpha \geq \frac{1}{2}\min(\beta, \gamma)$ and

i) $\left|\mathbb{E}[\widehat{P}_\ell - P]\right| \leq c_1\, 2^{-\alpha\,\ell}$

ii) $\mathbb{E}[\widehat{Y}_\ell] = \begin{cases} \mathbb{E}[\widehat{P}_0], & \ell = 0 \\ \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & \ell > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_\ell] \leq c_2\, N_\ell^{-1} 2^{-\beta\,\ell}$

iv) $\mathbb{E}[C_\ell] \leq c_3\, 2^{\gamma\,\ell}$

MLMC Theorem

then there exists a positive constant c_4 such that for any $\varepsilon < 1$ there exist L and N_ℓ for which the multilevel estimator

$$\widehat{Y} = \sum_{\ell=0}^{L} \widehat{Y}_\ell,$$

has a mean-square-error with bound $\mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$

with an expected computational cost C with bound

$$C \le \begin{cases} c_4\, \varepsilon^{-2}, & \beta > \gamma, \\ c_4\, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = \gamma, \\ c_4\, \varepsilon^{-2-(\gamma-\beta)/\alpha}, & 0 < \beta < \gamma. \end{cases}$$

MLMC Theorem

Two observations of optimality:

- MC simulation needs $O(\varepsilon^{-2})$ samples to achieve RMS accuracy ε. When $\beta > \gamma$, the cost is optimal — $O(1)$ cost per sample on average.

  (Would need multilevel QMC to further reduce costs)

- When $\beta < \gamma$, another interesting case is when $\beta = 2\alpha$, which corresponds to $\mathbb{E}[\widehat{Y}_\ell]$ and $\sqrt{\mathbb{E}[\widehat{Y}_\ell^2]}$ being of the same order as $\ell \to \infty$.

  In this case, the total cost is $O(\varepsilon^{-\gamma/\alpha})$, which is the cost of a single sample on the finest level — again optimal.

MLMC generalisation

The theorem is for scalar outputs P, but it can be generalised to multi-dimensional (or infinite-dimensional) outputs with

i) $\left\| \mathbb{E}[\widehat{P}_\ell - P] \right\| \le c_1 \, 2^{-\alpha \ell}$

ii) $\mathbb{E}[\widehat{Y}_\ell] = \begin{cases} \mathbb{E}[\widehat{P}_0], & \ell = 0 \\ \mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}], & \ell > 0 \end{cases}$

iii) $\mathbb{V}[\widehat{Y}_\ell] \equiv \mathbb{E}\left[\left\| \widehat{Y}_\ell - \mathbb{E}[\widehat{Y}_\ell] \right\|^2 \right] \le c_2 \, N_\ell^{-1} 2^{-\beta \ell}$

Original multilevel research by Heinrich in 1999 did this for parametric integration, estimating $g(\lambda) \equiv \mathbb{E}[f(x, \lambda)]$ for a finite-dimensional r.v. x.

MLMC Challenges

- not always obvious how to couple coarse and fine levels
  i.e. what does $\widehat{P}_\ell(\omega^{(n)}) - \widehat{P}_{\ell-1}(\omega^{(n)})$ mean?

- some creativity required to handle discontinuous functionals, where a small difference between the underlying coarse and fine simulations can produce an $O(1)$ difference in the output

- numerical analysis to determine the decay rate of V_ℓ can be tough

Financial application

- basket of 5 underlying assets, modelled by Geometric Brownian Motion

$$\mathrm{d}S_i = r\, S_i\, \mathrm{d}t + \sigma_i\, S_i\, \mathrm{d}W_i$$

  with correlation between 5 driving Brownian motions

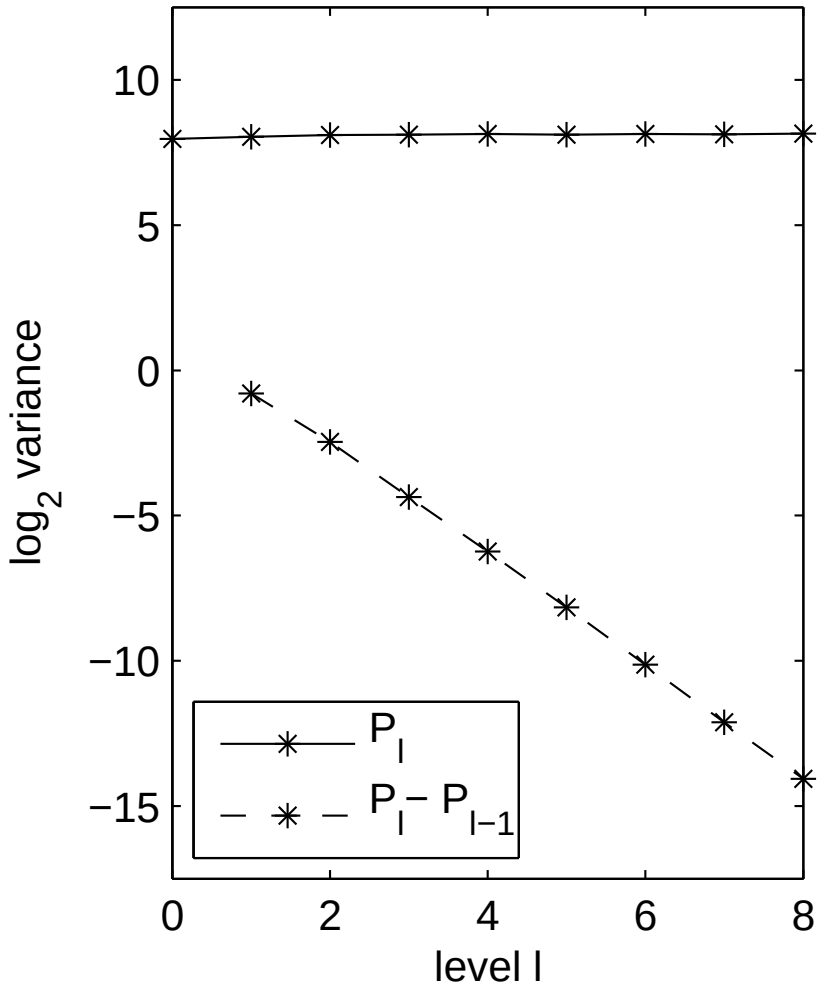
- Milstein numerical approximation

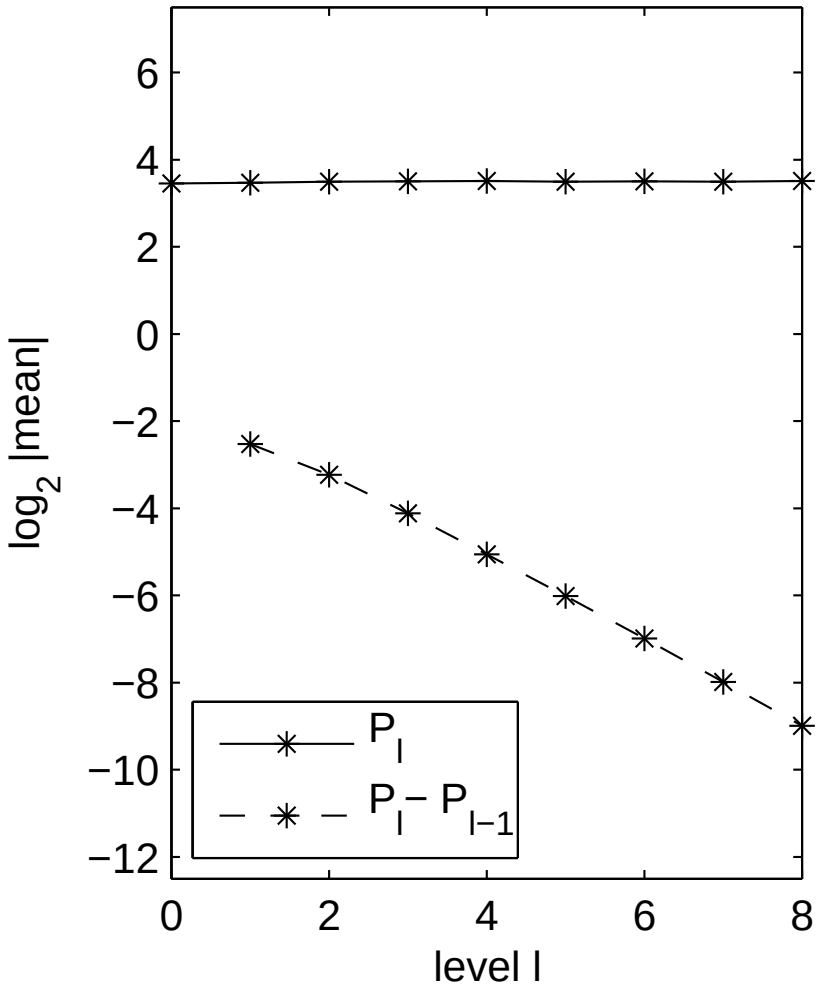
- standard call option is piecewise linear function of average at final time T

- digital call option is discontinuous function of average

Financial application

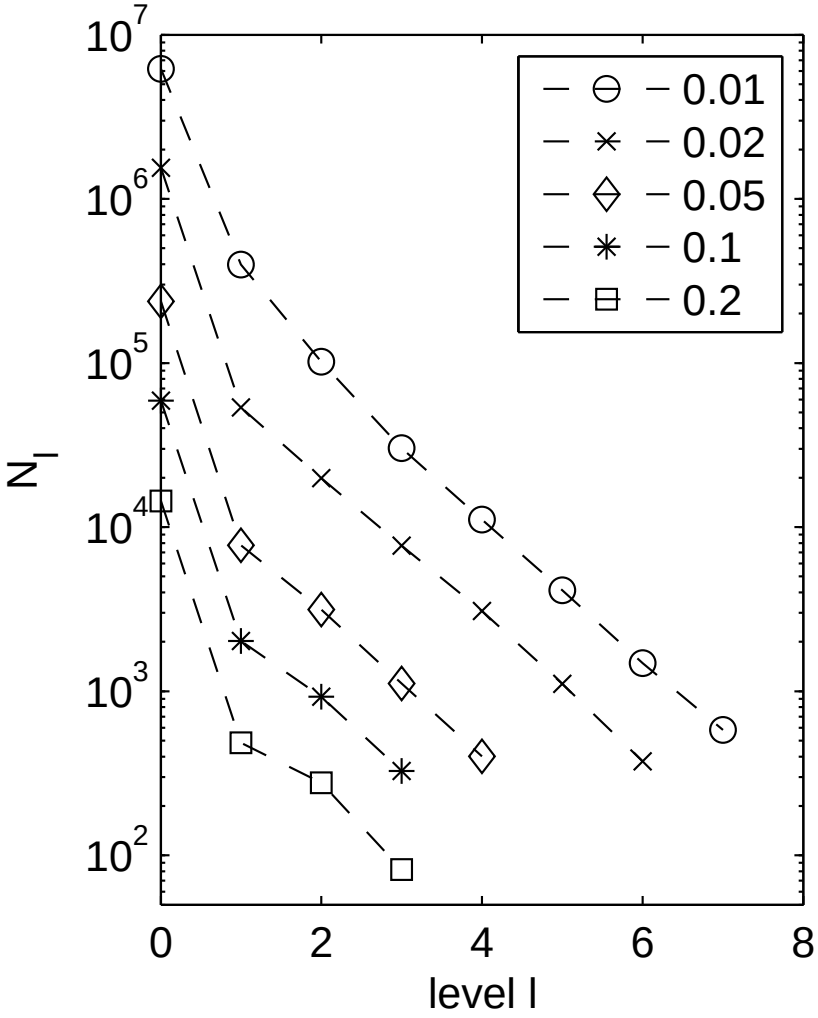
Standard call option:

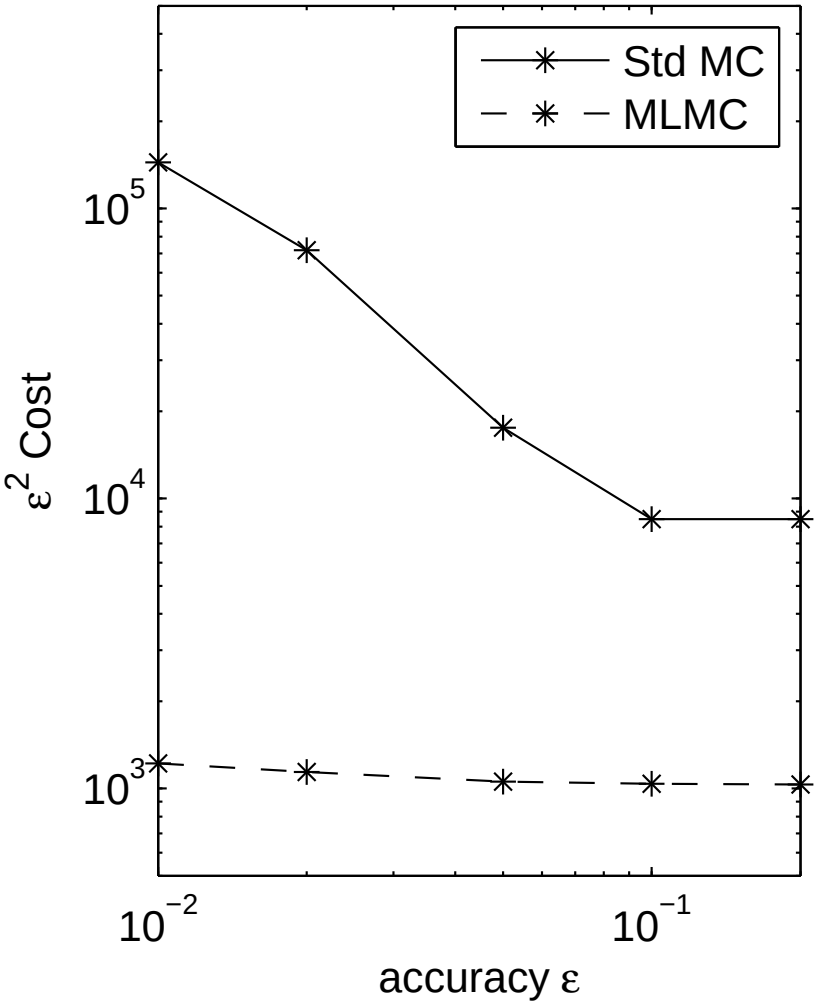




Financial application

Standard call option:





Digital options

In a digital option, the payoff is a discontinuous function of the final state.

Using the Milstein approximation, first order strong convergence means that $O(h_\ell)$ of the simulations have coarse and fine paths on opposite sides of a discontinuity.

Hence,

$$\widehat{P}_\ell - \widehat{P}_{\ell-1} = \begin{cases} O(1), & \text{with probability } O(h_\ell) \\ O(h_\ell), & \text{with probability } O(1) \end{cases}$$

so

$$\mathbb{E}[\widehat{P}_\ell - \widehat{P}_{\ell-1}] = O(h_\ell), \quad \mathbb{E}[(\widehat{P}_\ell - \widehat{P}_{\ell-1})^2] = O(h_\ell),$$

and hence $V_\ell = O(h_\ell)$, not $O(h_\ell^2)$

Digital options

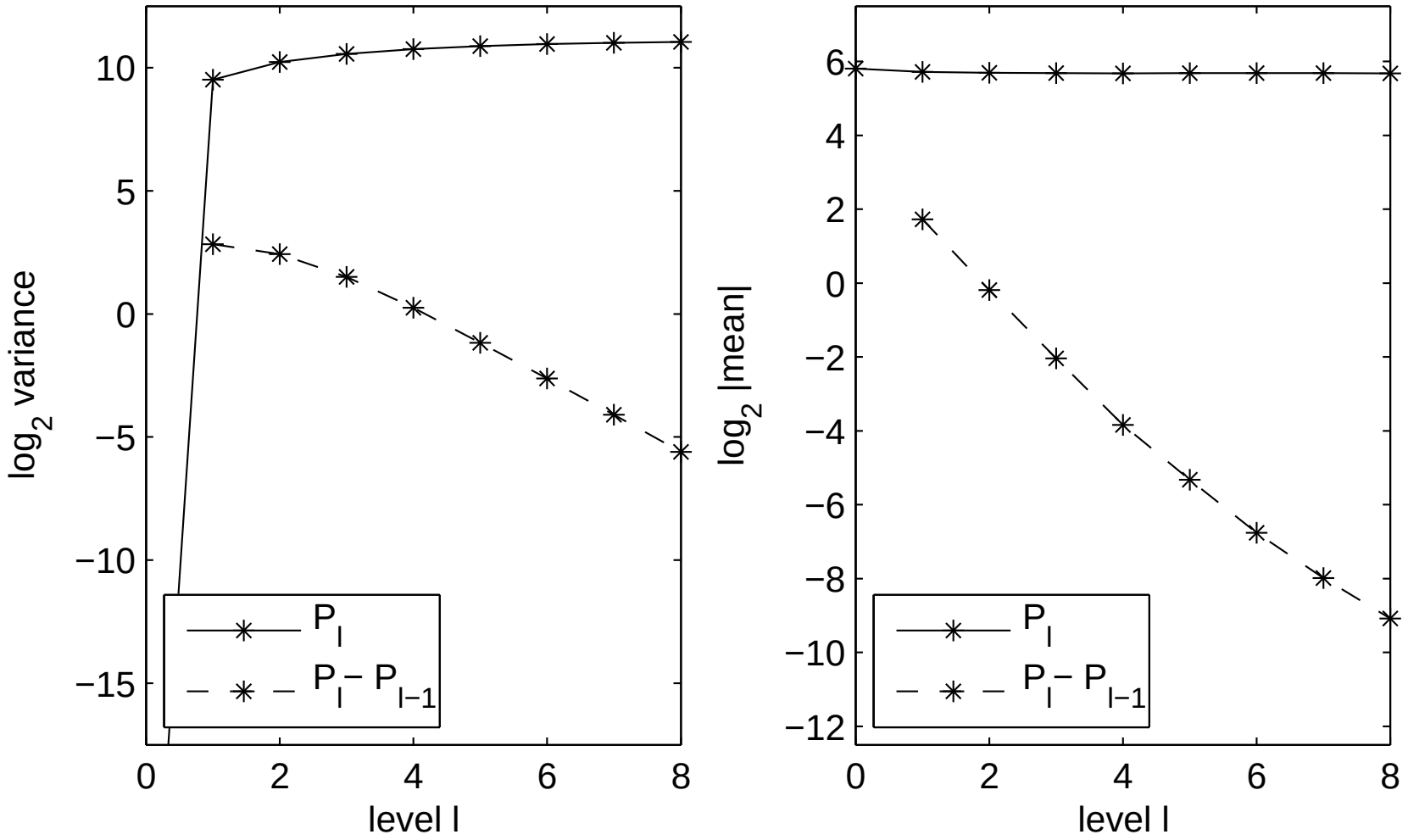
Three fixes:

- Conditional expectation: using the Euler discretisation instead of Milstein for the final timestep, conditional on all but the final Brownian increment, the final state has a Gaussian distribution, with a known analytic conditional expectation in simple cases
- Splitting: split each path simulation into M paths by trying M different values for the Brownian increment for the last fine path timestep
- Change of measure: when the expectation is not known, can use a change of measure so the coarse path takes the same final state as the fine path — difference in the "payoff" now comes from the Radon-Nikodym derivative

These all effectively smooth the payoff – end up with $V_\ell = O(h_\ell^{3/2})$.

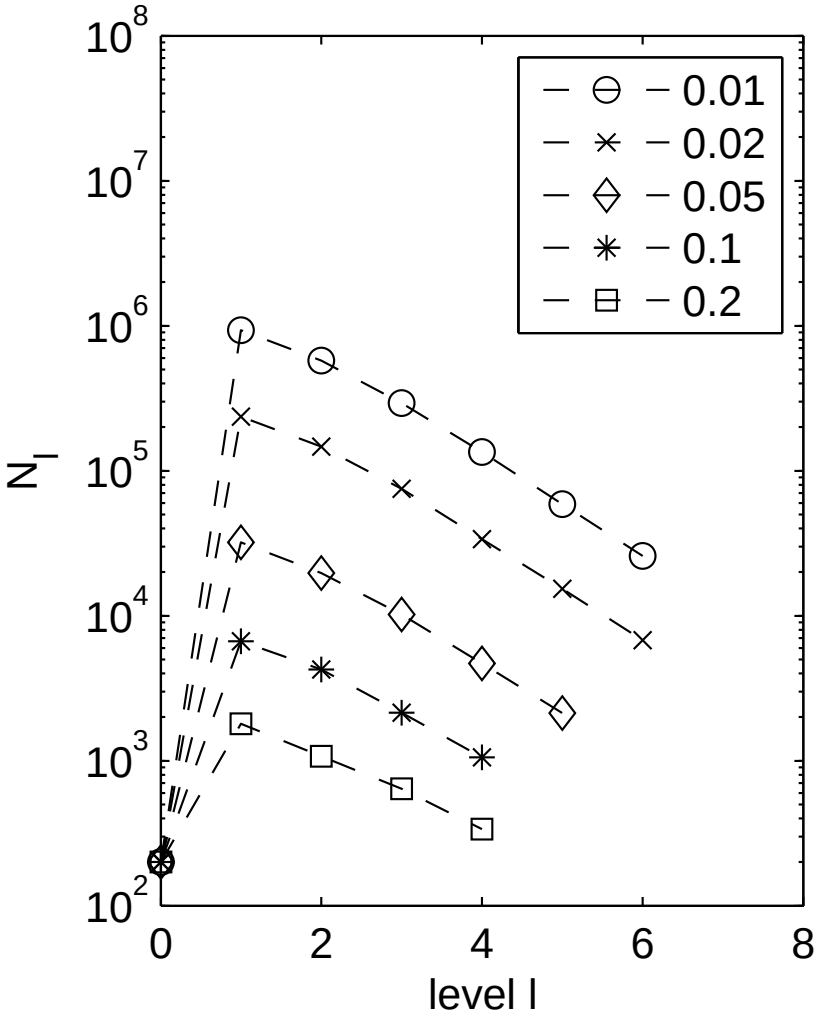
Financial application

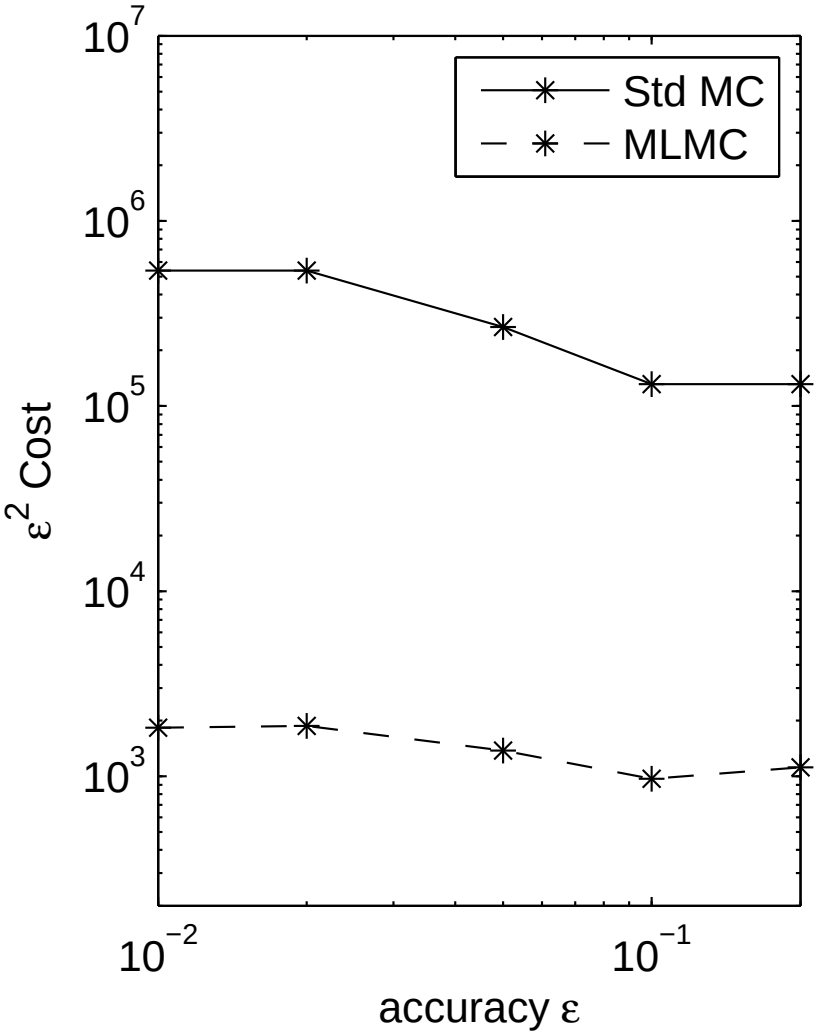
Digital call option:



Financial application

Digital call option:





Numerical Analysis

	Euler		Milstein	
option	numerics	analysis	numerics	analysis
Lipschitz	$O(h)$	$O(h)$	$O(h^2)$	$O(h^2)$
Asian	$O(h)$	$O(h)$	$O(h^2)$	$O(h^2)$
lookback	$O(h)$	$O(h)$	$O(h^2)$	$o(h^{2-\delta})$
barrier	$O(h^{1/2})$	$o(h^{1/2-\delta})$	$O(h^{3/2})$	$o(h^{3/2-\delta})$
digital	$O(h^{1/2})$	$O(h^{1/2}\lvert\log h\rvert)$	$O(h^{3/2})$	$o(h^{3/2-\delta})$

Table: V_ℓ convergence observed numerically (for GBM) and proved analytically (for more general SDEs)

Euler analysis due to G, Higham & Mao (2009) and Avikainen (2009).
Milstein analysis due to G, Debrabant & Rößler (2012).

Other MLMC work on SDEs

- financial sensitivities ("Greeks") – Burgos (2011)
- jump-diffusion models – Xia (2011)
- Lévy processes – Dereich (2010), Marxen (2010), Dereich & Heidenreich (2011), Xia (2013), Kyprianou (2014)
- American options – Belomestny & Schoenmakers (2011)
- Milstein in higher dimensions without Lévy areas – G. & Szpruch (2014)
- adaptive timesteps – Hoel, von Schwerin, Szepessy, Tempone (2012), G, Lester, Whittle (2014)

SPDEs

- quite natural application, with better cost savings than SDEs due to higher dimensionality

- range of applications
  - Graubner & Ritter (Darmstadt) – parabolic
  - G, Reisinger (Oxford) – parabolic
  - Cliffe, G, Scheichl, Teckentrup (Bath/Nottingham) – elliptic
  - Barth, Jenny, Lang, Meyer, Mishra, Müller, Schwab, Sukys, Zollinger (ETH Zürich) – elliptic, parabolic, hyperbolic
  - Harbrecht, Peters (Basel) – elliptic
  - Efendiev (Texas A&M) – numerical homogenization
  - Vidal-Codina, G, Peraire (MIT) – reduced basis approximation
  - G, Hou, Zhang (Caltech) – numerical homogenization

Engineering Uncertainty Quantification

Simplest possible example:

- 3D elliptic PDE, with uncertain boundary data
- grid spacing proportional to $2^{-\ell}$ on level ℓ
- cost is $O(2^{+3\ell})$, if using an efficient multigrid solver
- 2nd order accuracy means that

$$\widehat{P}_\ell(\omega) - P(\omega) \approx c(\omega)\, 2^{-2\ell}$$

$$\Longrightarrow \quad \widehat{P}_{\ell-1}(\omega) - \widehat{P}_\ell(\omega) \approx 3\, c(\omega)\, 2^{-2\ell}$$

- hence, $\alpha=2,\ \beta=4,\ \gamma=3$
- cost is $O(\varepsilon^{-2})$ to obtain ε RMS accuracy
- this compares to $O(\varepsilon^{-3/2})$ cost for one sample on finest level, so $O(\varepsilon^{-7/2})$ for standard Monte Carlo

Stochastic chemical reactions

In stochastic chemical reaction simulations, each reaction is a Poisson process with a rate which depends on the current concentrations.

In the "tau-leaping" method the reaction rates are frozen at the start of the timestep, so for each reaction need to sample from a Poisson process

$$P(\lambda\,\Delta t)$$

to determine the number of reactions in that timestep.

(As $\lambda\,\Delta t \to \infty$, the standard deviation becomes smaller relative to the mean, and it approaches the deterministic limit.)

Stochastic chemical reactions

Anderson & Higham (2011) have developed a very efficient multilevel version of this algorithm – big savings because finest level usually has 1000's of timesteps.

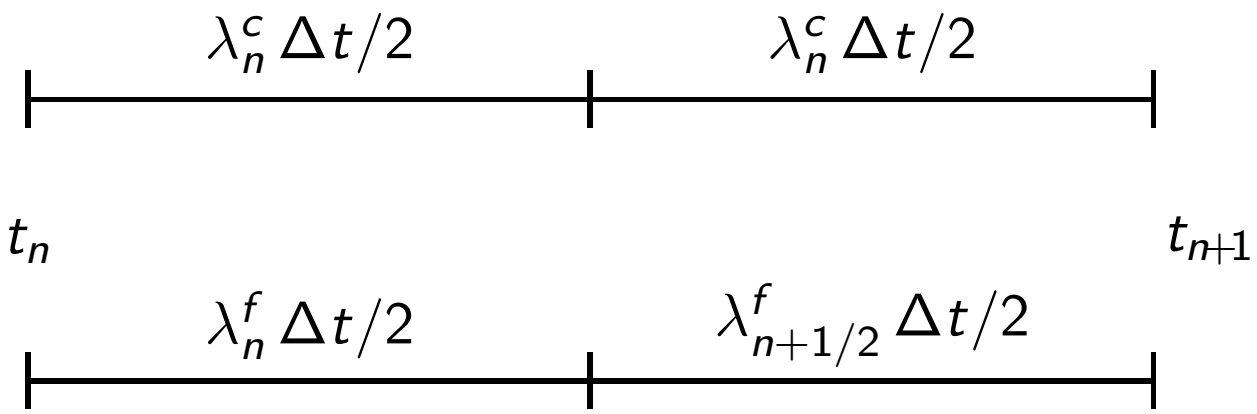
Key challenge: how to couple coarse and fine path simulations?

Crucial observation: $P(t_1) + P(t_2) \overset{d}{=} P(t_1 + t_2) \quad \text{for } t_1, t_2 \geq 0$

Stochastic chemical reactions

Solution:

- simulate the Poisson variable on the coarse timestep as the sum of two fine timestep Poisson variables
- couple the fine path and coarse path Poisson variables by using common variable based on smaller of two rates



If $\lambda_n^c > \lambda_n^f$, use $P(\lambda_n^c \Delta t/2) \sim P(\lambda_n^f \Delta t/2) + P((\lambda_n^c - \lambda_n^f)\Delta t/2)$

Other MLMC applications

- parametric integration, integral equations (Heinrich)
- multilevel QMC (Dick, G, Kuo, Scheichl, Schwab, Sloan)
- mixed precision computation on FPGAs (Korn, Ritter, Wehn)
- MLMC for MCMC (Scheichl, Schwab, Stuart, Teckentrup)
- Coulomb collisions in plasma (Caflisch)
- nested simulation (Haji-Ali & Tempone, Hambly & Reisinger)
- invariant distribution of contractive Markov process (Glynn & Rhee)
- invariant distribution of contractive SDEs (G, Lester & Whittle)

Three MLMC extensions

- unbiased estimation – Rhee & Glynn (2015)
  - randomly selects the level for each sample
  - no bias, and finite expected cost and variance if $\beta > \gamma$
- Richardson-Romberg extrapolation – Lemaire & Pagès (2013)
  - reduces the weak error, and hence the number of levels required
  - particularly helpful when $\beta < \gamma$
- Multi-Index Monte Carlo – Haji-Ali, Nobile, Tempone (2015)
  - important extension to MLMC approach, combining MLMC with sparse grid methods

Feynman-Kac project with Francisco Bernal

Suppose that X_t satisfies the SDE

$$\mathrm{d}X_t = a(X_t, t)\,\mathrm{d}t + b(X_t, t)\,\mathrm{d}W_t$$

in bounded domain D, where W_t is an uncorrelated Brownian motion, and let

$$u(x,t) = \mathbb{E}\left[\int_t^\tau E(t,s)\,f(X_s,s)\,\mathrm{d}s \;+\; E(t,\tau)\,g(X_\tau,\tau) \mid X_t = x\right]$$

where τ is the first time at which X_t crosses the boundary ∂D, and

$$E(t_0,t_1) = \exp\left(-\int_{t_0}^{t_1} V(X_t,t)\,\mathrm{d}t\right).$$

Feynman-Kac formula

If $f(x,t), g(x,t), V(x,t), a(x,t), b(x,t)$ are all Lipschitz continuous, then the Feynman-Kac formula states that $u(x,t)$ satisfies the PDE

$$\frac{\partial u}{\partial t} + \sum_j a_j \frac{\partial u}{\partial x_j} + \tfrac{1}{2} \sum_{j,k,l} b_{j,l} b_{k,l} \frac{\partial^2 u}{\partial x_j \partial x_k} - V\, u + f = 0$$

in domain D, subject to $u(x,t) = g(x,t)$ on the boundary ∂D.

Hence, can estimate the solution to a high-dimensional PDE at a particular point (x,t) through Monte Carlo simulation of the SDE.

This also extends to linear and nonlinear functionals of the PDE solution.

Numerical approximation

Let $\widehat{X}_t$ be the piecewise-constant Euler-Maruyama approximation and define

$$\widehat{E}(t_0, t_1) = \exp\left(-\int_{t_0}^{t_1} V(\widehat{X}_t, t)\,\mathrm{d}t\right),$$

and let

$$\widehat{u}(x,t) = \mathbb{E}\left[\int_t^{\widehat{\tau}} \widehat{E}(t,s)\, f(\widehat{X}_s, s)\,\mathrm{d}s + \widehat{E}(t,\widehat{\tau})\, g(\widehat{X}_{\widehat{\tau}}, \widehat{\tau}) \mid \widehat{X}_t = x\right],$$

with the Euler-Maruyama discretisation beginning at time t, and with $\widehat{\tau}$ being the exit time.

Numerical approximation

The Euler-Maruyama method has strong accuracy $O(h^{1/2})$, and the natural definition of τ gives an $O(h^{1/2})$ weak error too.

For standard Monte Carlo method, ε RMS accuracy needs $O(\varepsilon^{-2})$ paths, each with $h = O(\varepsilon^2)$, so total cost is $O(\varepsilon^{-4})$

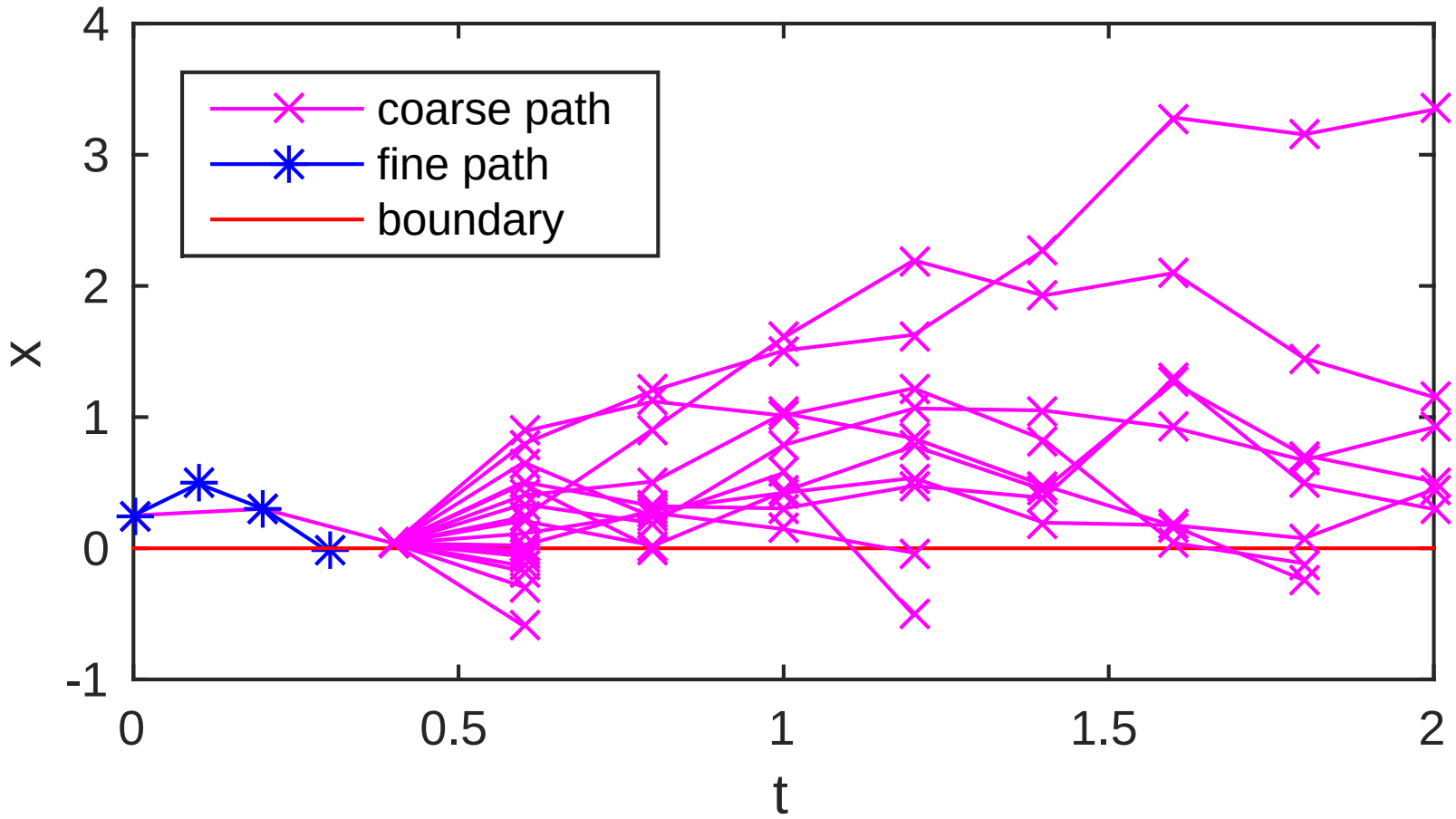
Gobet reduced this to $O(\varepsilon^{-3})$ by shifting the boundary by $O(h^{1/2})$ to improve the weak accuracy to $O(h)$.

Alternatively, Higham *et al* use MLMC to achieve $O(\varepsilon^{-3}|\log\varepsilon|^3)$ complexity without shifting the boundary.

MLMC challenge

When coarse or fine path exits the domain, the other is within $O(h^{1/2})$.

However, there is a $O(h^{1/2})$ probability that it will not exit the domain until much later $\Longrightarrow V_\ell = O(h^{1/2})$.



MLMC challenge

How can we do better?

Similar to previous work on digital options, split second path into multiple copies, and average their outputs to approximate the conditional expectation – see also recent work by Dickmann & Schweizer (2014).

$O(h^{1/2})$ expected time to exit for second path, so can afford to use $O(h^{-1/2})$ copies of second path.

This gives an approximation to the conditional expectation resulting in $\widehat{P}_\ell - \widehat{P}_{\ell-1} \approx O(h^{1/2})$, so $V_\ell \approx O(h_\ell)$.

Numerical results confirm this, and supporting numerical analysis is almost complete.

Conclusions

- multilevel idea is very simple; key question is how to apply it in new situations, and perform the numerical analysis
- discontinuous output functions can cause problems, but there is a lot of experience now in coping with this
- there are also "tricks" which can be used in situations with poor strong convergence
- being used for an increasingly wide range of applications; biggest computational savings when coarsest (reasonable) approximation is much cheaper than finest
- currently, getting at least 100$\times$ savings for SPDEs and stochastic chemical reaction simulations

References

Webpages for my research papers and talks:

`people.maths.ox.ac.uk/gilesm/mlmc.html`
`people.maths.ox.ac.uk/gilesm/slides.html`

Webpage for new 70-page *Acta Numerica* review and MATLAB test codes:

`people.maths.ox.ac.uk/gilesm/acta/`

– contains references to almost all MLMC research, including some very early related work by Achi Brandt

MLMC Community

Webpage: people.maths.ox.ac.uk/gilesm/mlmc_community.html

Abo Academi (Avikainen) – numerical analysis
Basel (Harbrecht) – elliptic SPDEs, sparse grids
Bath (Kyprianou, Scheichl, Shardlow, Yates) – elliptic SPDEs, MCMC, Lévy-driven SDEs, stochastic chemical modelling
Chalmers (Lang) – SPDEs
Duisburg (Belomestny) – Bermudan and American options
Edinburgh (Davie, Szpruch) – SDEs, numerical analysis
EPFL (Abdulle) – stiff SDEs and SPDEs
ETH Zürich (Jenny, Jentzen, Schwab) – SPDEs, multilevel QMC
Frankfurt (Gerstner, Kloeden) – numerical analysis, fractional Brownian motion
Fraunhofer ITWM (Iliev) – SPDEs in engineering
Hong Kong (Chen) – Brownian meanders, nested simulation in finance
IIT Chicago (Hickernell) – SDEs, infinite-dimensional integration, complexity analysis
Kaiserslautern (Heinrich, Korn, Ritter) – finance, SDEs, parametric integration, complexity analysis
KAUST (Tempone, von Schwerin) – adaptive time-stepping, stochastic chemical modelling
Kiel (Gnewuch) – randomized multilevel QMC
LPMA (Frikha, Lemaire, Pagès) – numerical analysis, multilevel extrapolation, finance applications
Mannheim (Neuenkirch) – numerical analysis, fractional Brownian motion
MIT (Peraire) – uncertainty quantification, SPDEs
Munich (Hutzenthaler) – numerical analysis
Oxford (Baker, Giles, Hambly, Reisinger) – SDEs, SPDEs, numerical analysis, finance applications, stochastic chemical modelling
Passau (Müller-Gronbach) – infinite-dimensional integration, complexity analysis
Stanford (Glynn) – numerical analysis, randomized multilevel
Strathclyde (Higham, Mao) – numerical analysis, exit times, stochastic chemical modelling
Stuttgart (Barth) – SPDEs
Texas A&M (Efendiev) – SPDEs in engineering
UCLA (Caflisch) – Coulomb collisions in physics
UNSW (Dick, Kuo, Sloan) – multilevel QMC
UTS (Baldeaux) – multilevel QMC
Warwick (Stuart, Teckentrup) – MCMC for SPDEs
WIAS (Friz, Schoenmakers) – rough paths, fractional Brownian motion, Bermudan options
Wisconsin (Anderson) – numerical analysis, stochastic chemical modelling