# Multilevel Monte Carlo for Basket Options

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# Generic Problem

Suppose we have a financial option based on multiple underlying assets, each of which satisfies an SDE with general drift and volatility terms:

$$\mathrm{d}S(t) = a(S,t)\,\mathrm{d}t + b(S,t)\,\mathrm{d}W(t)$$

Will simulate these using the Milstein scheme:

$$\widehat{S}_{n+1} = \widehat{S}_n + a\,h + b\,\Delta W_n + \tfrac{1}{2}\,b'\,b\left((\Delta W_n)^2 - h\right)$$

– first order weak and strong convergence

# Standard MC Approach

Mean Square Error is $O\left(N^{-1} + h^2\right)$

- first term comes from variance of estimator
- second term comes from bias due to weak convergence

To make this $O(\varepsilon^2)$ requires

$$N = O(\varepsilon^{-2}), \quad h = O(\varepsilon) \quad \Longrightarrow \quad \text{cost} = O(N\,h^{-1}) = O(\varepsilon^{-3})$$

Aim is to improve this to $O\left(\varepsilon^{-2}\right)$, by combining simulations with different numbers of timesteps

# Multilevel MC Approach

Consider multiple sets of simulations with different timesteps $h_l = 2^{-l}\,T,\ l=0,1,\ldots,L$, and payoff $\widehat{P}_l$

$$\mathbb{E}[\widehat{P}_L] = \mathbb{E}[\widehat{P}_0] + \sum_{l=1}^{L} \mathbb{E}[\widehat{P}_l - \widehat{P}_{l-1}]$$

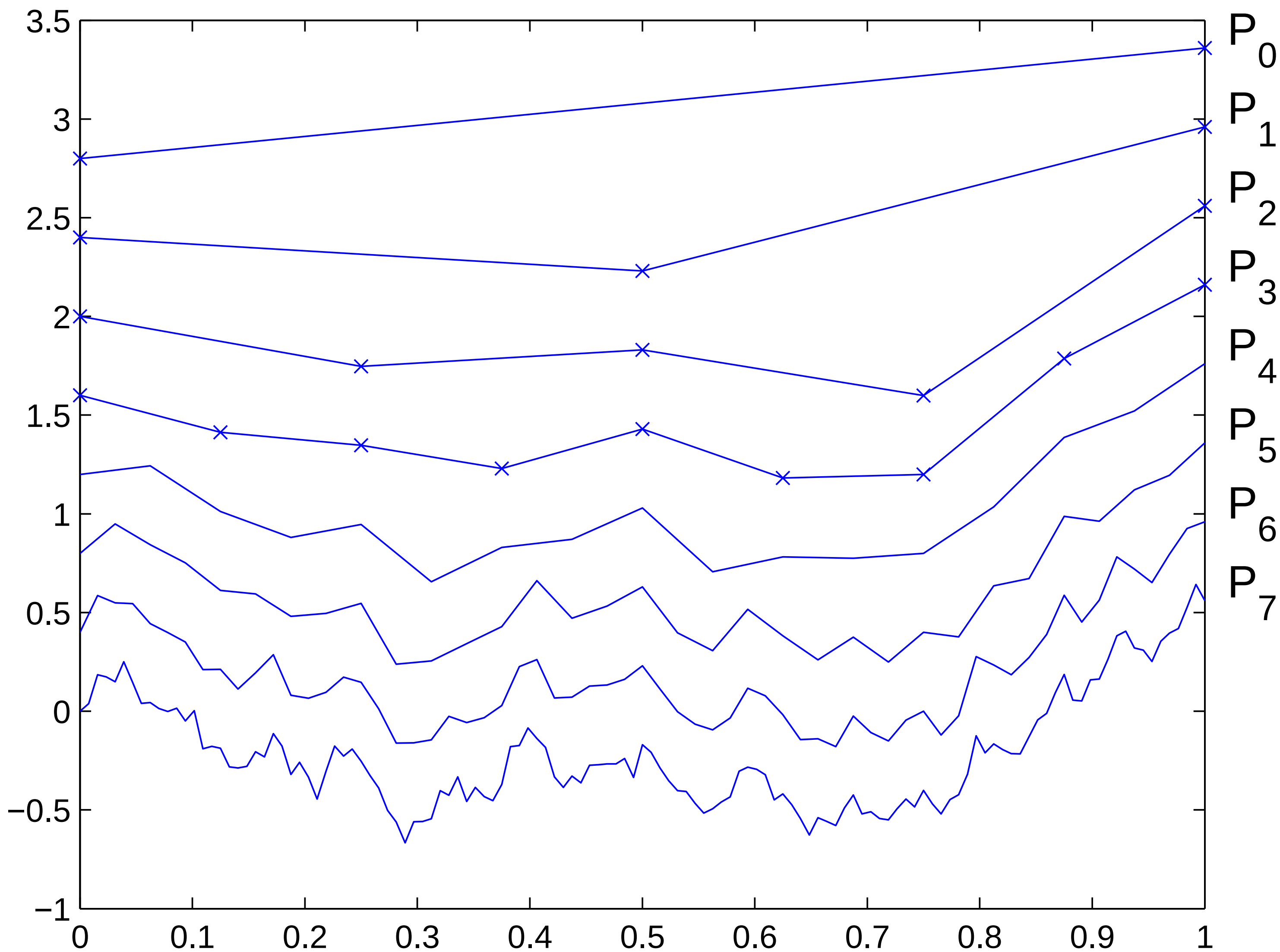
Expected value is same – aim is to reduce variance of estimator for a fixed computational cost.

Key point: approximate $\mathbb{E}[\widehat{P}_l - \widehat{P}_{l-1}]$ using $N_l$ simulations with $\widehat{P}_l$ and $\widehat{P}_{l-1}$ obtained using <u>same</u> Brownian path.

$$\widehat{Y}_l = N_l^{-1} \sum_{i=1}^{N_l} \left( \widehat{P}_l^{(i)} - \widehat{P}_{l-1}^{(i)} \right)$$

# Multilevel MC Approach

Discrete Brownian path at different levels



# Multilevel MC Approach

Using independent paths for each level, the variance of the combined estimator is

$$\mathbb{V}\left[\sum_{l=0}^{L} \widehat{Y}_l\right] = \sum_{l=0}^{L} N_l^{-1} V_l, \qquad V_l \equiv \mathbb{V}[\widehat{P}_l - \widehat{P}_{l-1}],$$

and the computational cost is proportional to $\displaystyle\sum_{l=0}^{L} N_l\, h_l^{-1}$.

Hence, the variance is minimised for a fixed computational cost by choosing $N_l$ to be proportional to $\sqrt{V_l\, h_l}$.

The constant of proportionality can be chosen so that the combined variance is $O(\varepsilon^2)$.

# Multilevel MC Approach

For the Milstein discretisation and a European option with a Lipschitz payoff function

$$\mathbb{V}[\widehat{P}_l - P] = O(h_l^2) \quad \Longrightarrow \quad \mathbb{V}[\widehat{P}_l - \widehat{P}_{l-1}] = O(h_l^2)$$

and the optimal $N_l$ is asymptotically proportional to $h_l^{3/2}$.

To make the combined variance $O(\varepsilon^2)$ requires

$$N_l = O(\varepsilon^{-2} h_l^{3/2})$$

and hence we obtain an $O(\varepsilon^2)$ MSE for a computational cost which is $O(\varepsilon^{-2})$.

# Results

Basket of 5 underlying assets, each GBM with

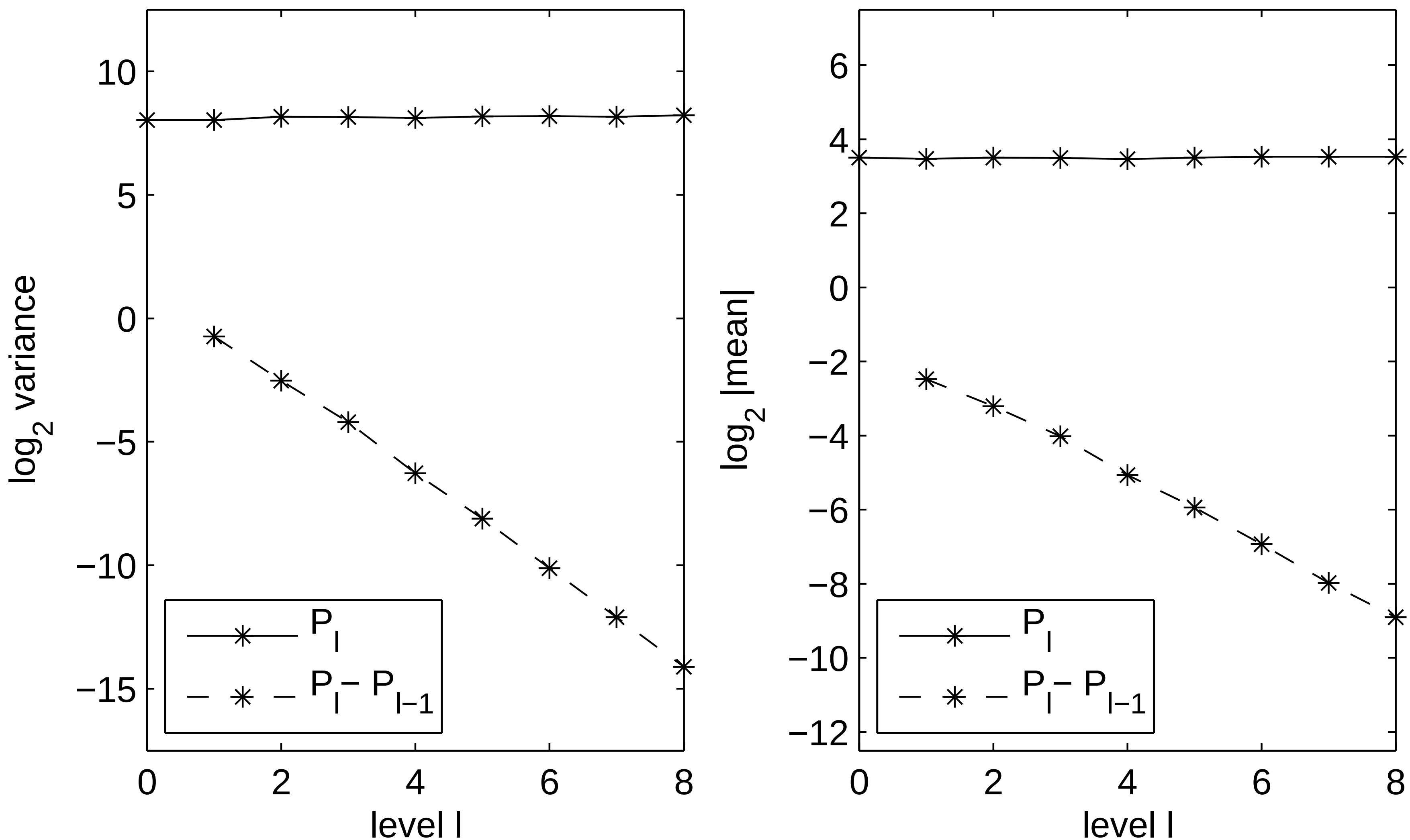
$$r = 0.05, \;\; T = 1, \;\; S_i(0) = 100, \;\; \sigma = (0.2,\, 0.25,\, 0.3,\, 0.35,\, 0.4),$$

and correlation $\rho = 0.25$ between each of the driving Brownian motions.

All options are based on arithmetic average $\overline{S}$ of 5 assets, with strike $K = 100$ (and barrier $B = 85$).

# MLMC Results

European call, $\exp(-rT)\ \max(\overline{S}(T) - K, 0)$



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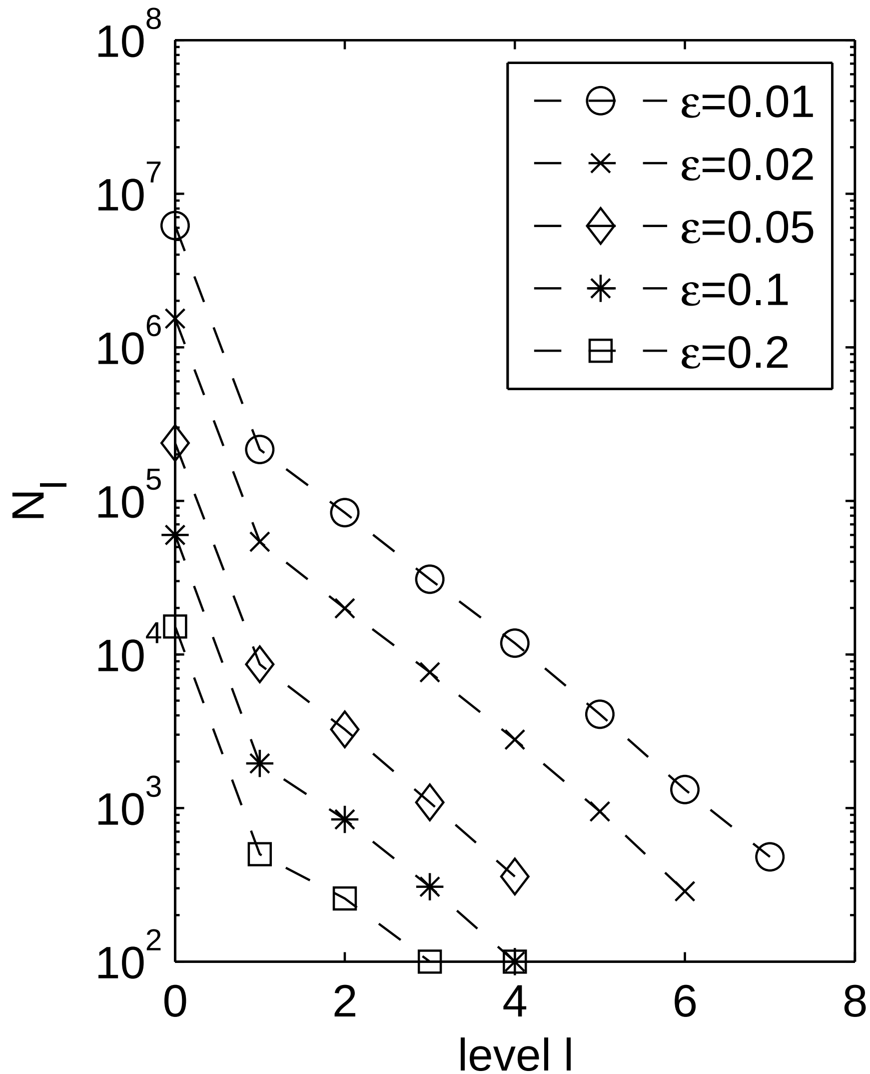
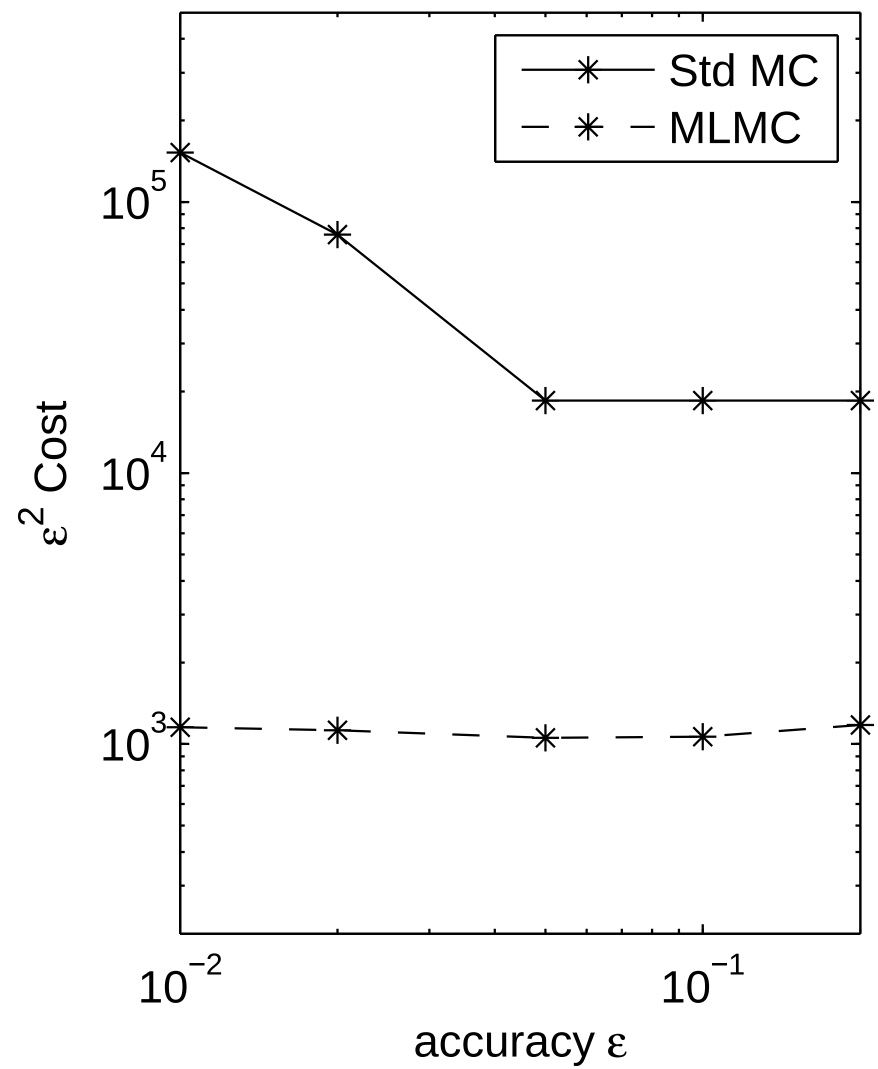


# MLMC Approach

**Theorem:** *Let $P$ be a functional of the solution of one or more SDEs, and $\widehat{P}_l$ the discrete approximation using a timestep $h_l = 2^{-l}\, T$.*

**If** *there exist independent estimators $\widehat{Y}_l$ based on $N_l$ Monte Carlo samples, with computational complexity (cost) $C_l$, and positive constants $\alpha \geq \frac{1}{2}, \beta, c_1, c_2, c_3$ such that*

*i)* $\left| \mathbb{E}[\widehat{P}_l - P] \right| \leq c_1\, h_l^{\alpha}$

*ii)* $\mathbb{E}[\widehat{Y}_l] = \begin{cases} \mathbb{E}[\widehat{P}_0], & l=0 \\ \mathbb{E}[\widehat{P}_l - \widehat{P}_{l-1}], & l>0 \end{cases}$

*iii)* $\mathbb{V}[\widehat{Y}_l] \leq c_2\, N_l^{-1} h_l^{\beta}$

*iv)* $C_l \leq c_3\, N_l\, h_l^{-1}$

# Multilevel MC Approach

**then** *there exists a positive constant* $c_4$ *such that for any* $\varepsilon < e^{-1}$ *there are values* $L$ *and* $N_l$ *for which the multilevel estimator*

$$\widehat{Y} = \sum_{l=0}^{L} \widehat{Y}_l,$$

*has Mean Square Error* $MSE \equiv \mathbb{E}\left[\left(\widehat{Y} - \mathbb{E}[P]\right)^2\right] < \varepsilon^2$

*with a computational complexity* $C$ *with bound*

$$C \leq \begin{cases} c_4\, \varepsilon^{-2}, & \beta > 1, \\ c_4\, \varepsilon^{-2} (\log \varepsilon)^2, & \beta = 1, \\ c_4\, \varepsilon^{-2-(1-\beta)/\alpha}, & 0 < \beta < 1. \end{cases}$$

# Milstein Scheme

Other options:

- lookback and barrier, based on min/max
- digital, with discontinuous payoff

require careful construction of the multilevel estimator using Brownian interpolation / extrapolation

Extends naturally to basket options based on a weighted average of underlying assets

# Milstein Scheme

Brownian interpolation: within each timestep, model the behaviour as simple Brownian motion conditional on the two end-points

$$\begin{aligned}\widehat{S}(t) &= \widehat{S}_n + \lambda(t)(\widehat{S}_{n+1} - \widehat{S}_n) \\ &\quad + b_n \Big( W(t) - W_n - \lambda(t)(W_{n+1} - W_n) \Big),\end{aligned}$$

where

$$\lambda(t) = \frac{t - t_n}{t_{n+1} - t_n}$$

There then exist analytic results for the distribution of the min/max/average over each timestep, and probability of crossing a barrier.

# Milstein Scheme

Brownian extrapolation for final timestep:

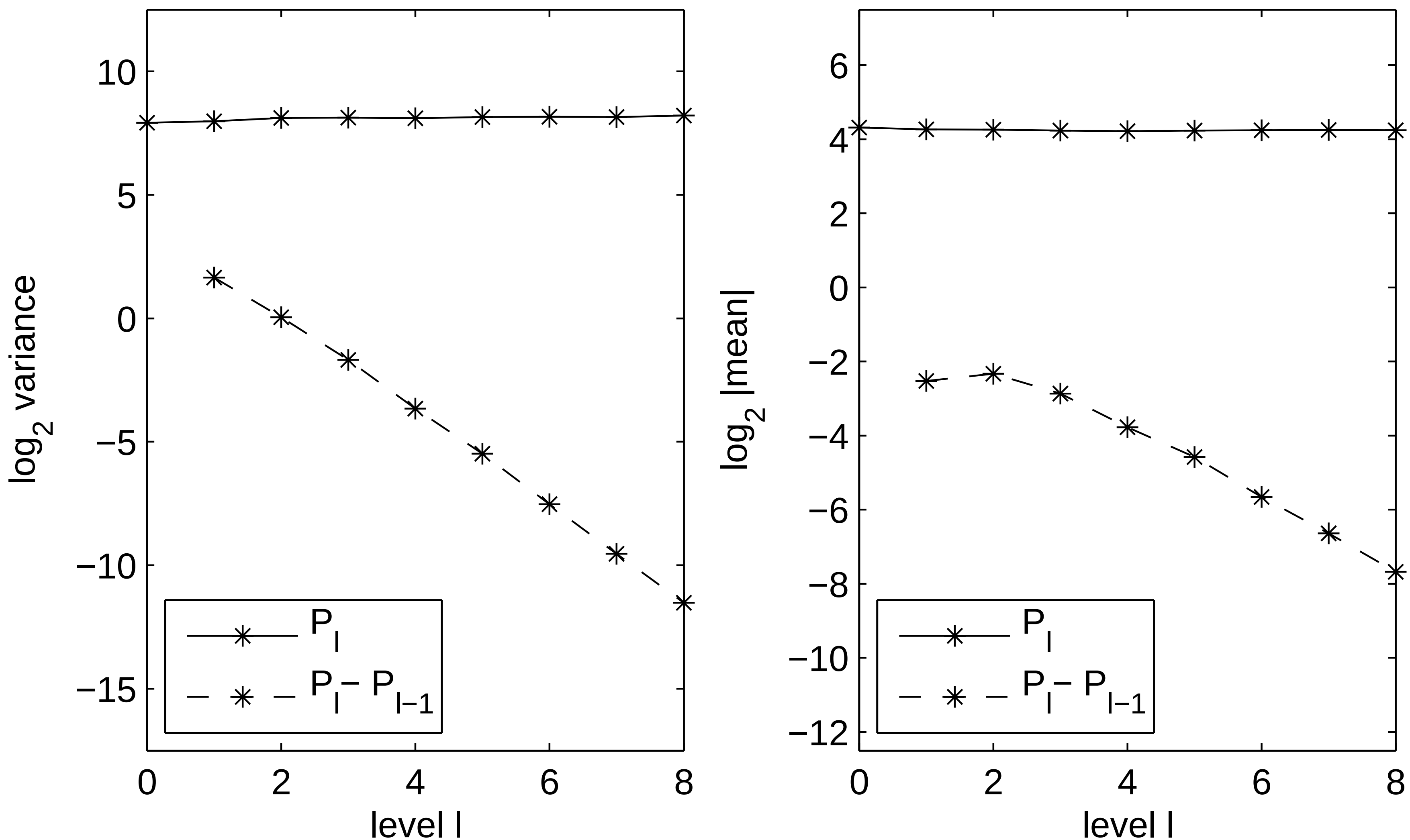
$$\widehat{S}_N = \widehat{S}_{N-1} + a_{N-1} h + b_{N-1} \Delta W_N$$

Considering all possible $\Delta W_N$ gives Gaussian distribution, for which a digital option has a known conditional expectation – example in Glasserman's book of payoff smoothing to allow pathwise calculation of Greeks.

Interpolation and extrapolation both work also for basket options based on a weighted average, since the average has a similar distribution.

# MLMC Results

Lookback option, $\exp(-rT)\,(\overline{S}(T) - \min_{0<t<T} \overline{S}(t))$



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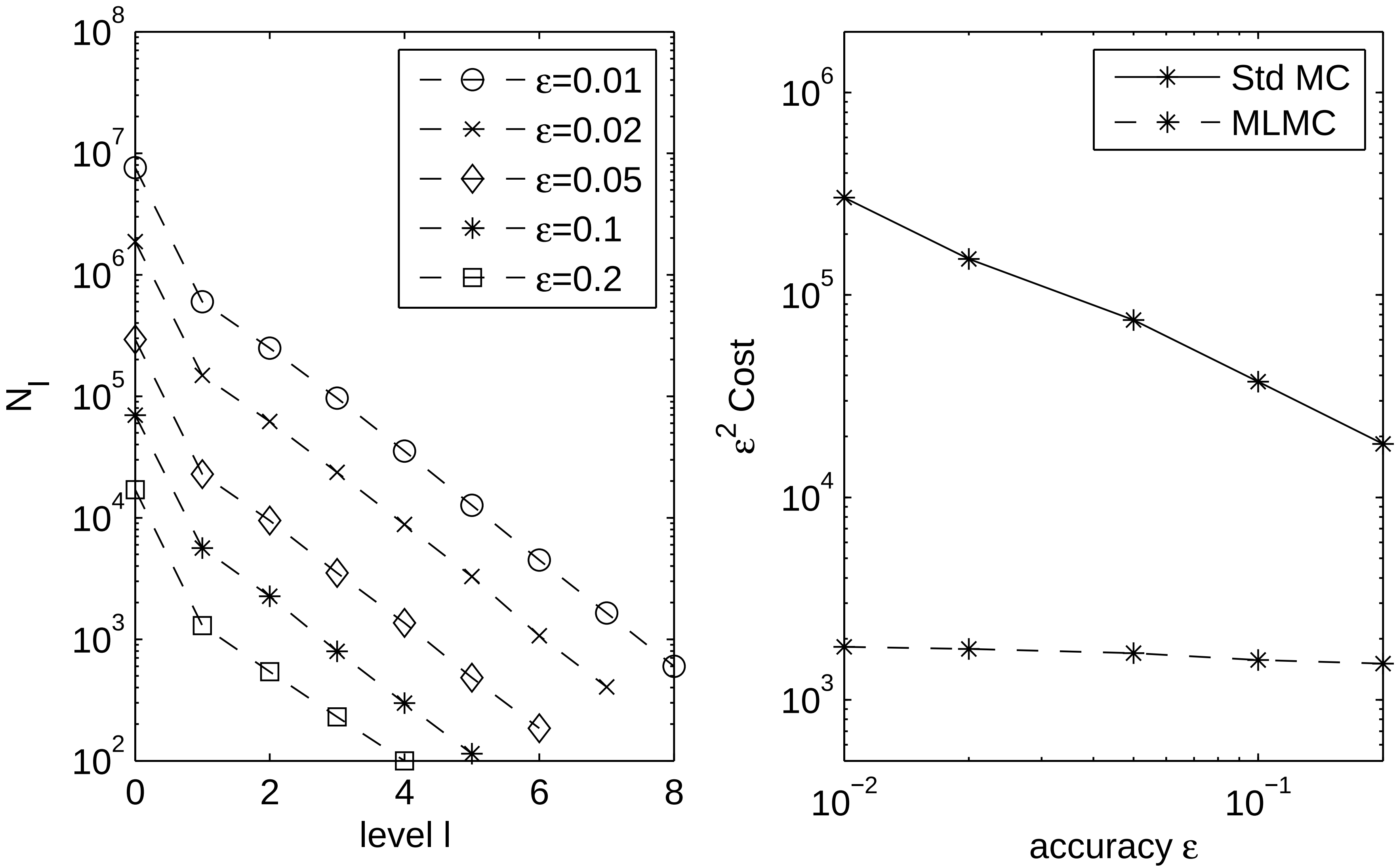
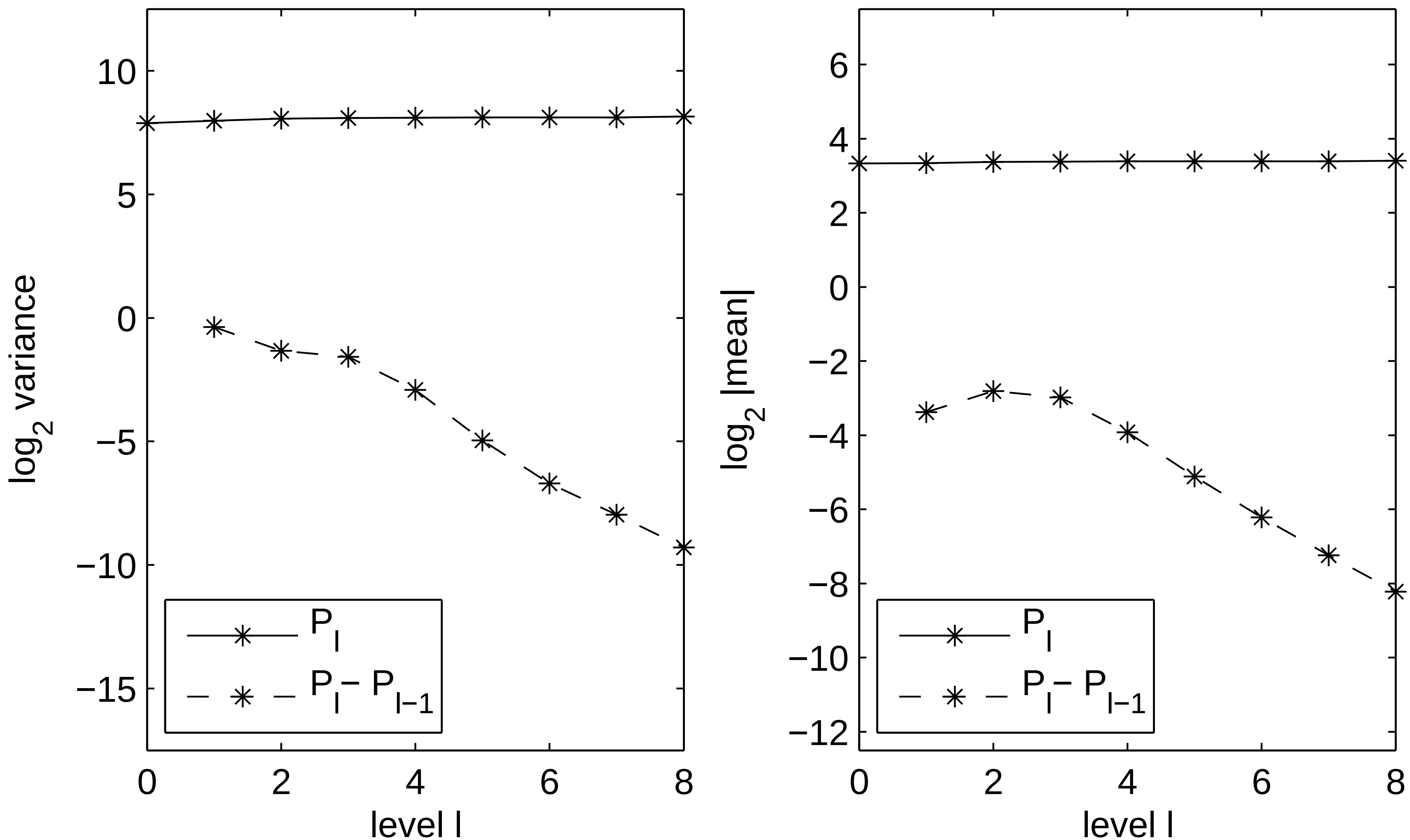


# MLMC Results

Barrier option, $\exp(-rT)\ \max(\overline{S}(T)-K,0)\ \mathbf{1}_{\min_{0<t<T} \overline{S}(t)>B}$



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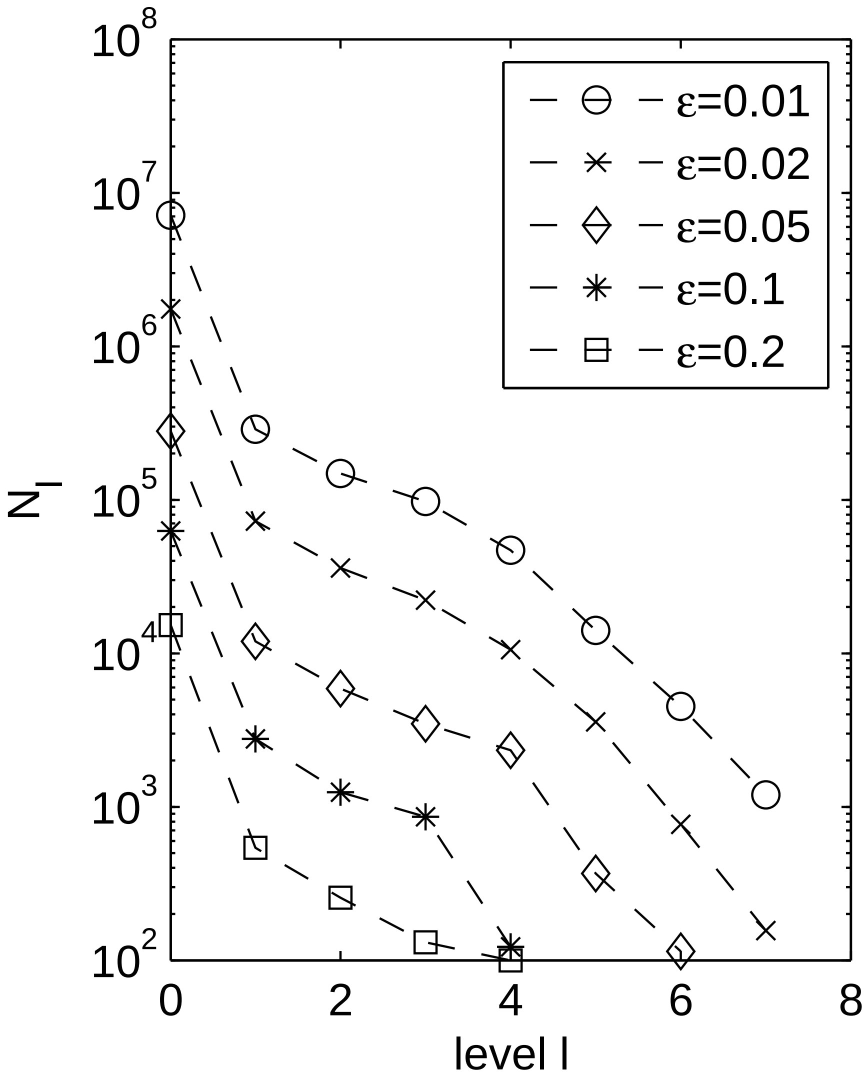
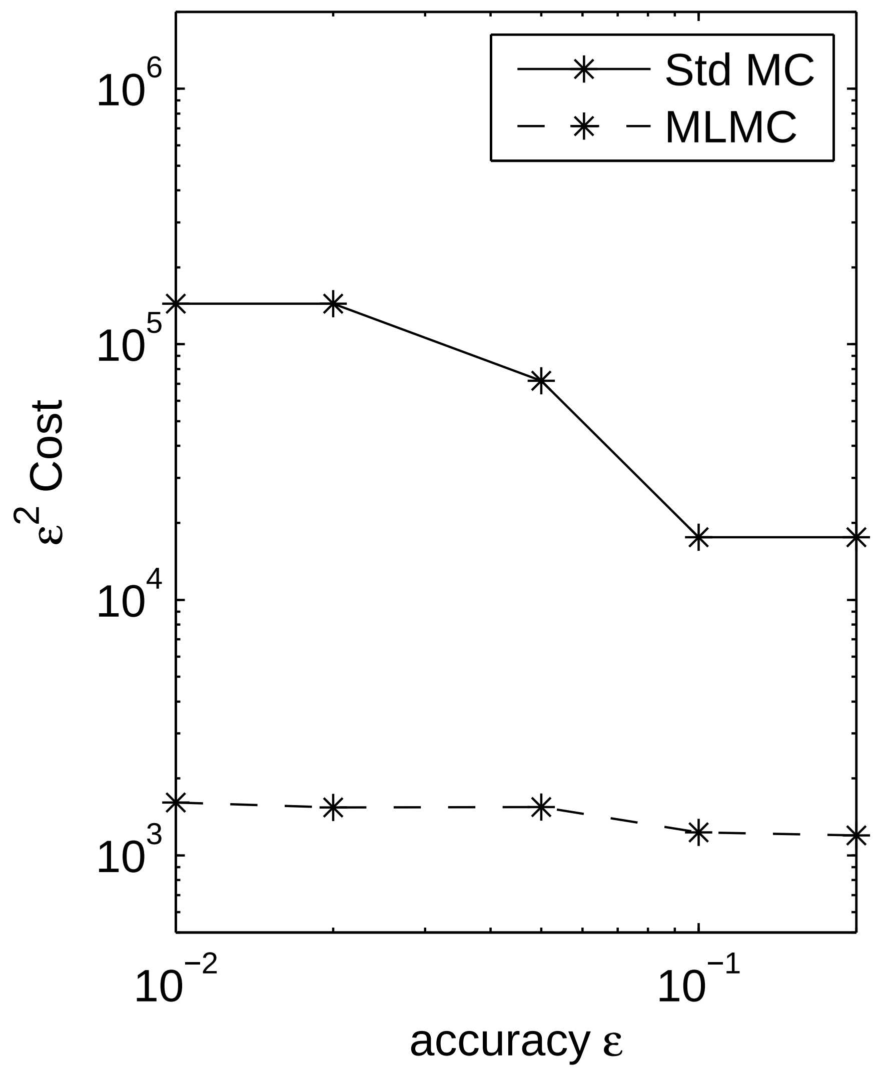
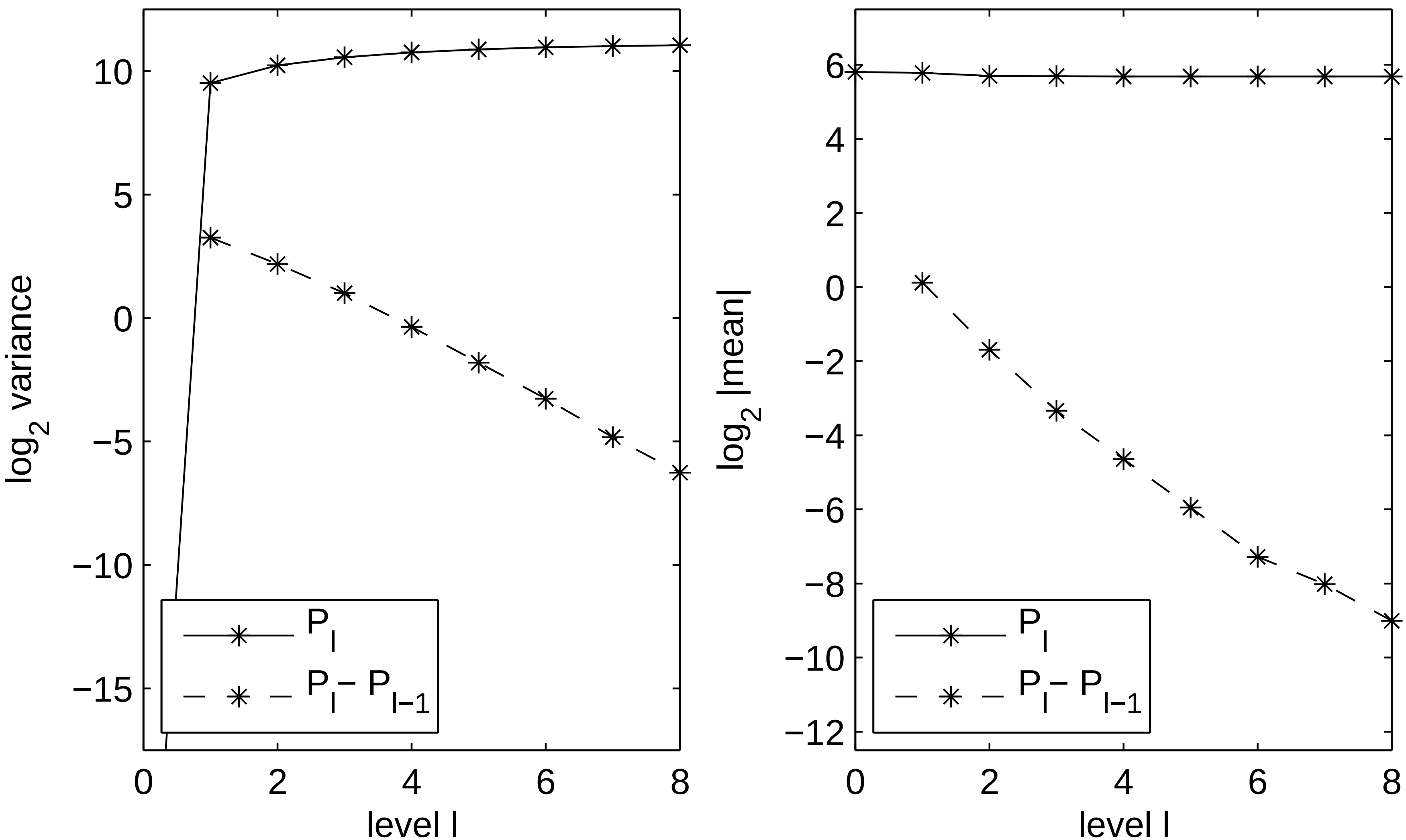


# MLMC Results

Digital option, $K\,\exp(-rT)\,\mathbf{1}_{\overline{S}(T)>K}$



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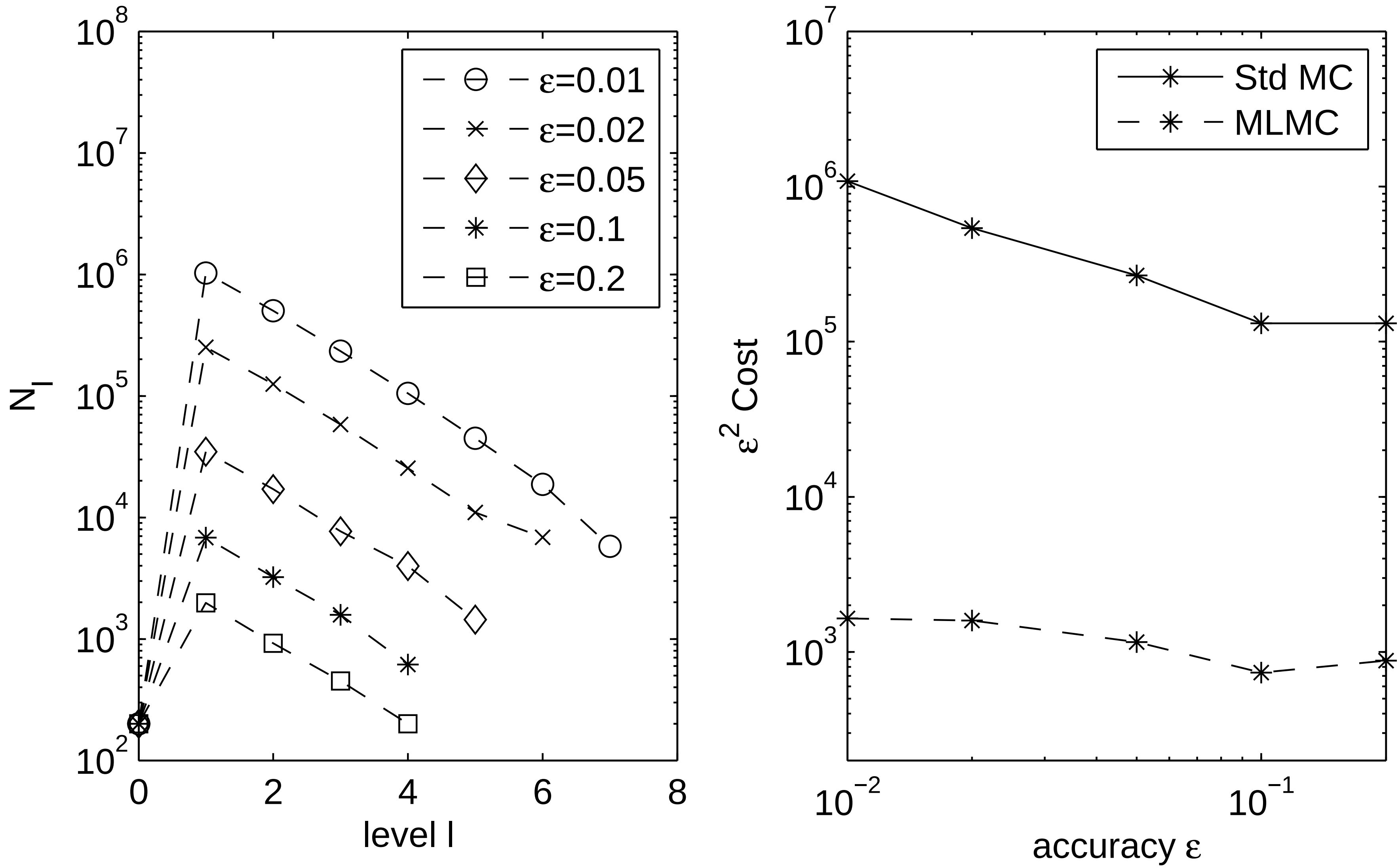


# Other basket options

More general basket options, not based on a simple weighted average, are more challenging.

For example, consider a European option which is a general, discontinuous function of the multiple underlying assets

The extrapolation approach produces a multivariate Gaussian as the conditional distribution at maturity, but in most cases there is no simple expression for the conditional expected payoff

Most successful approach so far is to use splitting, using multiple samples for the final timestep to get an estimate for the conditional expectation

# Splitting

If $W$ and $Z$ are independent random variables, then for any function $g(W,Z)$ the estimator

$$\widehat{Y}_{M,N} = N^{-1} \sum_{n=1}^{N} \left( M^{-1} \sum_{m=1}^{M} g(W^{(n)}, Z^{(m,n)}) \right)$$

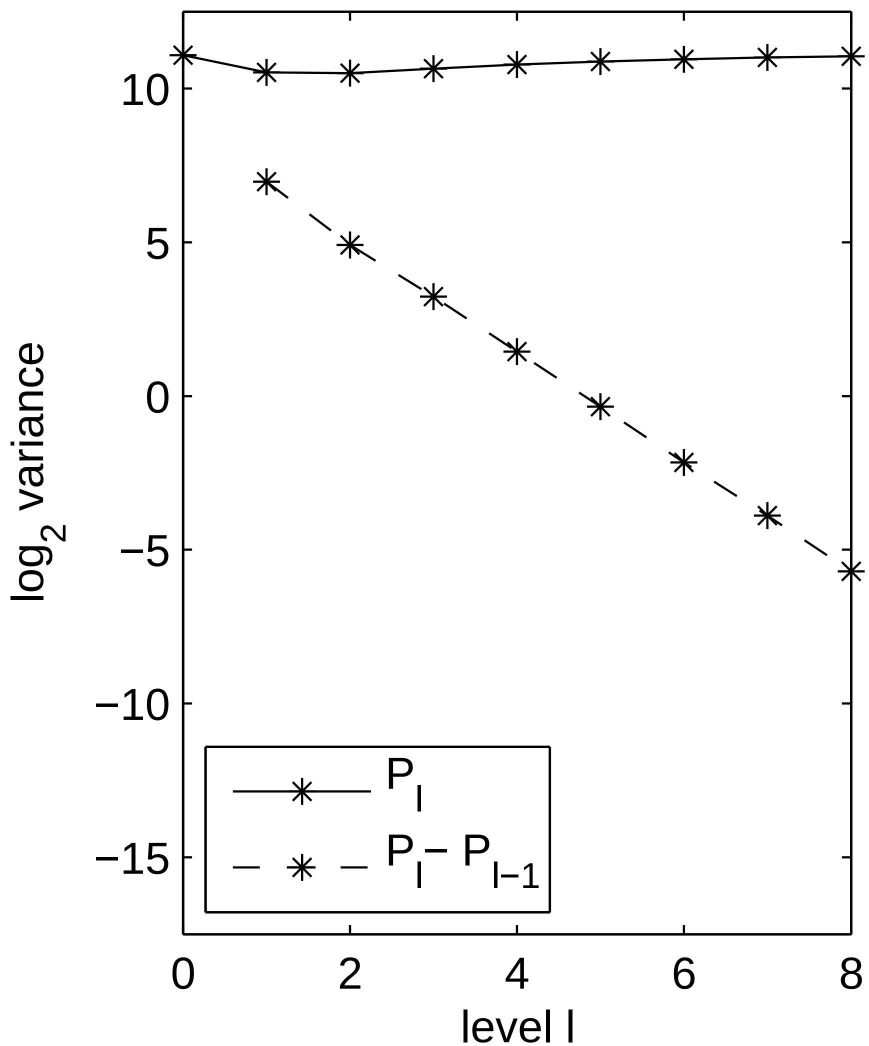
with independent samples $W^{(n)}$ and $Z^{(m,n)}$ is an unbiased estimator for $\mathbb{E}_{W,Z}\left[g(W,Z)\right] \equiv \mathbb{E}_W \left[ \mathbb{E}_Z[g(W,Z) \,|\, W] \right]$,
and its variance is

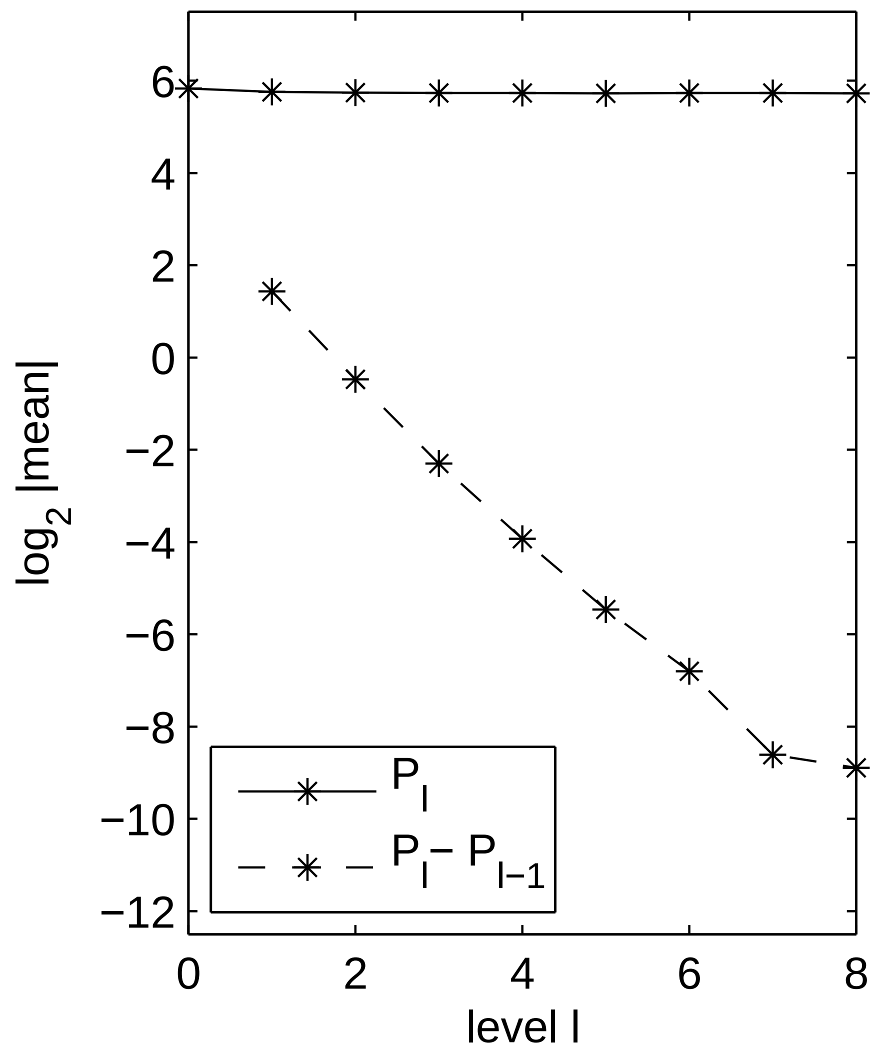
$$N^{-1}\, \mathbb{V}_W \left[ \mathbb{E}_Z[g(W,Z) \,|\, W] \right] + (MN)^{-1}\, \mathbb{E}_W \left[ \mathbb{V}_Z[g(W,Z) \,|\, W] \right]$$

Can use estimates of variance and computational cost to determine optimal splitting

# Splitting

Digital option, $K\ \exp(-rT)\, \mathbf{1}_{\overline{S}(T)>K}$



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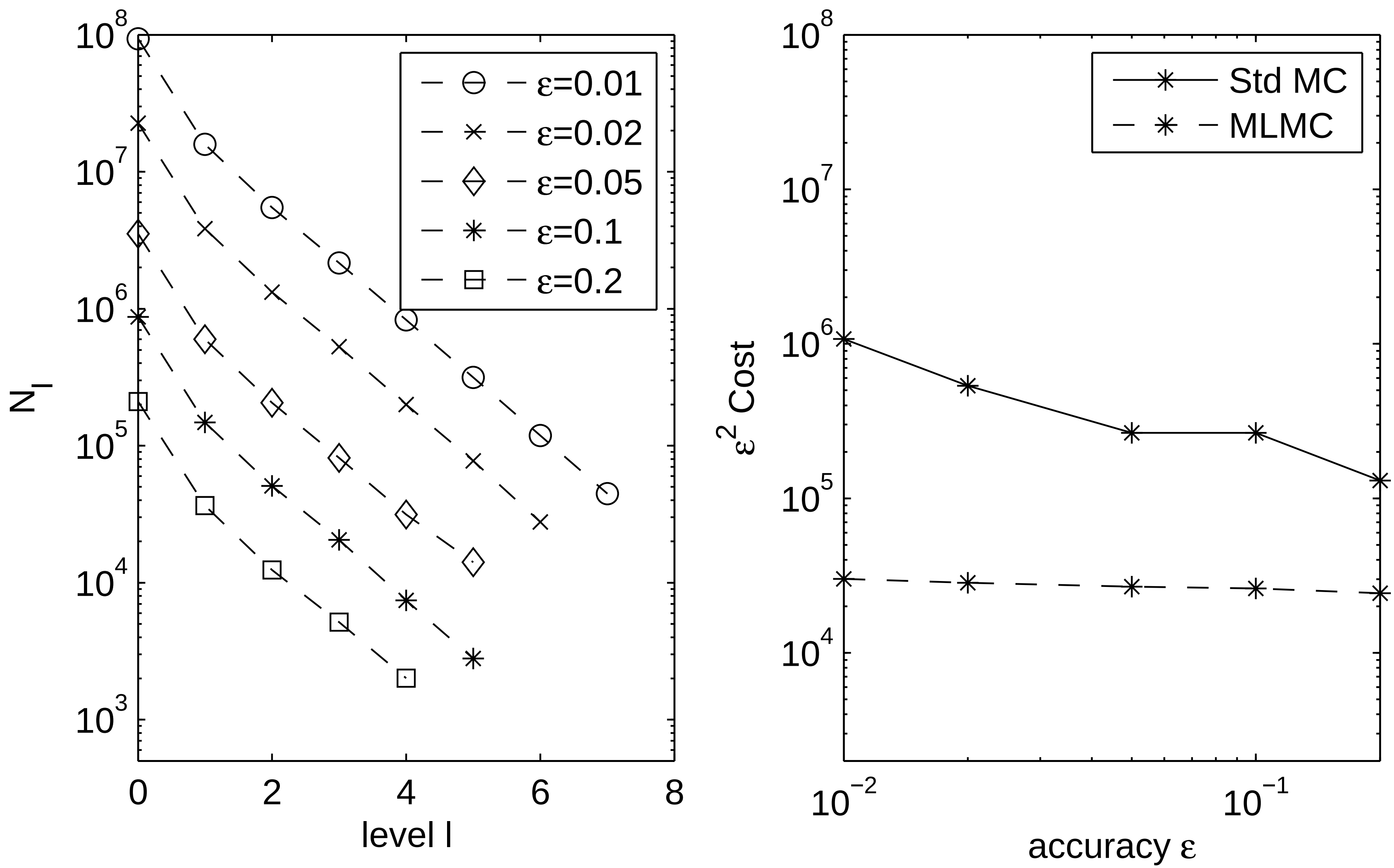
Digital option, $K\ \exp(-rT)\ \mathbf{1}_{\overline{S}(T)>K}$



# Conclusions

Multilevel Monte Carlo method has been successfully extended to basket options

- works best for European options with Lipschitz payoff, or more complex options on weighted average
- splitting is useful for general digital options

M.B. Giles, “Multilevel Monte Carlo path simulation”, *Operations Research*, 56(3):607-617, 2008.

M.B. Giles. “Improved multilevel Monte Carlo convergence using the Milstein scheme”, pp. 343-358 in *Monte Carlo and Quasi-Monte Carlo Methods 2006*, Springer, 2007.

Papers are available from:
`www.maths.ox.ac.uk/∼gilesm/finance.html`