In this set of questions the hat symbol denotes the Fourier transform on $\mathbb{Z}/N\mathbb{Z}$.

1. Suppose that $A \subset \mathbb{Z}/N\mathbb{Z}$ is an arithmetic progression. Show that

$$\sum_{r \in \mathbb{Z}/N\mathbb{Z}} |\hat{1}_A(r)| \leqslant C \log N,$$

where C is an absolute constant.

2. A set A in some abelian group is said to be a *Sidon set* if the only solutions to the equation $x + y = z + w$ with $x, y, z, w \in A$ are the trivial solutions in which $\{w, z\} = \{x, y\}$. Show that two Sidon sets of the same size are 2-isomorphic. Show that the set $\{(x, x^2) : x \in \mathbb{Z}/p\mathbb{Z}\}$ is a Sidon set in $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and deduce that for any N there is a Sidon subset of $\{1, \dots, N\}$ of size at least $c\sqrt{N}$, for some absolute constant $c > 0$.

3. Let A be a finite subset of $\mathbb{R}^n$, and let $s \geqslant 2$ be an integer. Show that A is Freiman s-isomorphic to a subset of $\mathbb{Z}$.

4. Suppose that $A \subset \mathbb{Z}/N\mathbb{Z}$ is a set of size $\lfloor N/2 \rfloor$, and that $|\hat{1}_A(r)| \leqslant N^{-c}$ whenever $r \neq 0$, where c is some absolute constant. Show that if $N > N_0(c)$ is large enough then A intersects every arithmetic progression P in $\mathbb{Z}/N\mathbb{Z}$ of length at least $N/100$.

5. Let N be prime. Let $R \subset \mathbb{Z}/N\mathbb{Z}$ be a set of size d, and suppose that $0 < \varepsilon < 1$. Prove the following statements about the Bohr set $B(R, \varepsilon)$:

(i) $|B(R, \varepsilon)| \geqslant \varepsilon^d N$;

(ii) $|B(R, \varepsilon)| \geqslant 4^{-d} |B(R, 2\varepsilon)|$;

(iii) $B(R, 2\varepsilon)$ can be covered by 10^d translates of $B(R, \varepsilon)$.

6. Show that there is a function $F : \mathbb{N} \to \mathbb{N}$ with the following property: any set $A \subset \mathbb{Z}$ of size n is Freiman 2-isomorphic to a subset of $\{1, \dots, F(n)\}$. Show that F must grow at least exponentially in n.

7. Suppose that $A \subset \mathbb{Z}$ is a set of size n. Show that there is a set $A' \subset A$, $|A'| \geqslant n/2s$, which admits an injective Freiman s-homomorphism into $[n]$.

8. Show that every set $A \subset \mathbb{Z}$ of size n contains a set of size at least $ne^{-c\sqrt{\log n}}$ which is free of 3-term arithmetic progressions. Show that A contains a Sidon set (cf. Q2) of size at least $c\sqrt{n}$.

9. Let p be a large prime, and suppose that $A \subset \mathbb{Z}/p\mathbb{Z}$ is a set of size at most $100 \log p$. Show that A is Freiman 2-isomorphic to a set of integers. *Is the same true for sets of size $100 \log p$?

10. Given a finite set $A \subset \mathbb{Z}$, define $\dim_s(A)$ to be the dimension of the space of Frieman s-homomorphisms from A to $\mathbb{Q}$, considered as a vector space over $\mathbb{Q}$. Show that if A is a random subset of $[n]$ (choosing each element independently at random with probability 1/2) then with probability tending to 1 as $n \to \infty$ we have $\dim_s(A) = 2$, for each fixed s.

11. Suppose that N is a prime, and let $f : \mathbb{Z}/N\mathbb{Z} \to \{-1, 1\}$ be a function.

(i) Show that there is at least one value of r such that the discrete Fourier coefficient $\hat{f}(r)$ has $|\hat{f}(r)| \geqslant N^{-1/2}$.

(ii) Show that if $f(x) = (x|N)$, the Legendre symbol, then $|\hat{f}(r)| = N^{-1/2}$ for all r.

(iii) Deduce that the same is true if $f(x) = \pm(x+a|n)$, for any fixed $A \in \mathbb{Z}/N\mathbb{Z}$ and for either choice of sign $\pm$.

(iv) *Prove the converse: that is, if $|\hat{f}(r)| = N^{-1/2}$ for all r, then f has the form given in (iii).

ben.green@maths.ox.ac.uk