CHAPTER 8

The sum-product phenomenon in \( \mathbb{C} \)

In this section we give some extremely elegant and (in retrospect!) simple arguments of Solymosi establishing sum-product phenomena in fields with a metric structure. We focus on \( \mathbb{R} \) and \( \mathbb{C} \).

**Theorem 8.1** (Solymosi). Suppose that \( A \) is a finite set of complex numbers. Then \( |A + A| + |A \cdot A| \geq c|A|^{5/4} \).

Essentially the only property of the field \( \mathbb{C} \) that is relevant for Solymosi’s argument is the so-called Besicovitch property.

**Definition 8.1** (Besicovitch constant). Suppose that \((X, d)\) is a metric space. The Besicovitch constant of \( X \) (if it is defined) is the largest \( k \) such that there exist balls \( B_i = B(x_i, r_i), i = 1, \ldots, k \) with the property that \( x_i \) is never in the interior of \( B_j \) if \( i \neq j \), and such that \( \bigcap_{i=1}^{k} B_i \) is nonempty.

**Lemma 8.1** (Besicovitch constant of \( \mathbb{C} \)). The Besicovitch constant of \( \mathbb{C} \) is 6.

**Proof.** This is simple Euclidean geometry exercise. Suppose that \( B_i = B(x_i, r_i), i = 1, \ldots, 7 \), are balls intersecting in some point \( z \). Suppose that the centres \( x_1, \ldots, x_7 \) are arranged in order, radially about \( z \). The angles \( x_1zx_2, x_2zx_3, \ldots, x_7zx_1 \) must be at least \( \pi/3 \) since the distance \( |x_i - x_{i+1}| \) is greater than or equal to both \( |x_i - z| \) and \( |x_{i+1} - z| \). This is obviously a contradiction.

**Proof of Theorem 8.1.** Suppose that \( A \subseteq \mathbb{C} \) is a finite set and that the additive doubling \( \sigma_+ [A] \) and the multiplicative doubling \( \sigma_\times [A] \) are both at most \( K \). Our aim is to show that \( K \geq c|A|^{1/4} \).

To each point \( a \in A \) associate the nearest neighbour \( a^* \) of \( a \) in \( A \setminus \{a\} \), making an arbitrary choice if there are ties to be broken. To motivate the proof, suppose that the following (false) assumption held: for any triple \((a_1, a_2, a_3) \in A \times A \times A\) the unique nearest neighbour to \( a_1 + a_2 \) in \( A + A \) is \( a_1^* + a_2 \), and the unique nearest neighbour to \( a_1a_2 \) in \( A \cdot A \) is \( a_1^*a_3 \). We could then consider the map
\[
\psi : A \times A \times A \to (A + A) \times (A + A) \times (A \cdot A) \times (A \cdot A)
\]
defined by \( \phi(a_1, a_2, a_3) = (a_1 + a_2, a_1^* + a_2, a_1a_3, a_1^*a_3) \). Now it is an easy algebraic exercise to see that this map is injective. Furthermore, by our false assumption, knowledge of \( a_1 + a_2 \) and \( a_1a_3 \) tells us the values of \( a_1^* + a_2 \) and \( a_1^*a_3 \), which means that \( \text{im}(\phi) \leq |A + A||A \cdot A| \). We would then have \( |A + A||A \cdot A| \geq |A|^3 \), a much stronger result than the one we have claimed.
The problem, of course, is our false assumption. It turns out that something a little weaker is true: for many triples \((a_1, a_2, a_3)\) there are not many points of \(A + A\) closer to \(a_1 + a_2\) than \(a_1^* + a_2\), and not many points of \(A \cdot A\) closer to \(a_1 a_3\) than \(a_1^* a_3\).

More precisely we will examine well-behaved triples \((a_1, a_2, a_3)\) for which \(a_1^* + a_2\) is “almost” the nearest neighbour of \(a_1 + a_2\) in \(A + A\) in the sense that

\[
U_{a_1, a_2} := |\{u \in A + A : |a_1^* + a_2 - u| \leq |(a_1^* + a_2) - (a_1 + a_2)|\}| \leq 100K
\]

and for which \(a_1^* a_3\) is “almost” the nearest neighbour of \(a_1 a_3\) in the sense that

\[
V(a_1, a_3) := |\{v \in A \cdot A : |a_1^* a_3 - v| \leq |a_1^* a_3 - a_1 a_3|\}| \leq 100K.
\]

It is not obvious that there are such triples, but we claim that this good behaviour is quite generic: there are at least \(|A|^3/2\) such triples.

Examining (8.1) in the first instance, fix \(a_2\). Then the balls \(B_{|a_1^* - a_1|}(a_1 + a_2),\ a_1 \in A\), have Besicovitch’s intersection property. It follows that no \(u\) can lie in 7 of them. It follows that

\[
\sum_{a_1} U_{a_1, a_2} \leq 6|A + A| \leq 6K|A|.
\]

An essentially identical argument using (8.2) implies that

\[
\sum_{a_1} V_{a_1, a_3} \leq 6K|A|.
\]

The number of pairs \((a_1, a_2)\) for which \(U_{a_1, a_2} \geq 100K\) is thus at most \(|A|^2/10\), as is the number of pairs \((a_1, a_3)\) for which \(V_{a_1, a_3} \geq 100K\). The claim follows.

Now suppose that \(x = a_1 + a_2, y = a_1^* + a_2, z = a_1 a_3\) and \(w = a_1^* a_3\) are known. The same simple algebraic exercise as before confirms that \(a_1, a_1^*, a_2, a_3\) may be recovered from knowledge of \(x, y, z\) and \(w\), and hence by the claim just proved the number of choices for the quadruple \((x, y, z, w)\) such that \((a_1, a_2, a_3)\) constitute a well-behaved triple is at least \(|A|^3/2\). Now there are \(|A + A|\) ways to specify \(x\) and \(|A \cdot A|\) ways to specify \(z\). Suppose these have been chosen, and consider the possible choices of \(y\). Single out one \(\overline{y}\) corresponding to a well-behaved triple \((\overline{a}_1, \overline{a}_2, \overline{a}_3)\) with \(|x - y| = |\overline{a}_1^* - \overline{a}_1|\) maximal. Then for all permissible \(y\) we have

\[
|\overline{a}_1 + \overline{a}_2 - y| = |x - y| \leq |x - \overline{y}| = |\overline{a}_1^* - \overline{a}_1|.
\]

By the definition of well-behaved triple, and specifically in view of (8.1), there are at most \(100K\) choices for \(y\). Similarly there are at most \(100K\) choices for \(w\). It follows that

\[
|A|^3/2 \leq |A + A| \cdot |A \cdot A| \cdot (100K)^2 \leq 10^4 K^4 |A|^2,
\]

from which the result follows immediately.